Answer on Question #51468 - Math – Statistics and Probability

The density function of a continuous random variable X is given by

$$f(x) = \begin{cases} Ae^{-x}, & 0 \le x \le 1 \\ 0, & otherwise \end{cases}$$

Evaluate the following:

a) Mean

b) Variance

c) $E[(2+3x)^2]$

<u>Solution</u>

Since $f(x)$ is a density function, it must satisfy the following relation

$$\int_0^1 f(x)dx = 1$$

Therefore

$$\int_0^1 Ae^{-x}dx = A(1-e^{-1}) = 1 \Rightarrow A = \frac{1}{1-e^{-1}} = \frac{e}{e-1}$$

a) Mean

$$E[X] = \mu = \int_0^1 xf(x)dx = \int_0^1 xAe^{-x}dx = -A\int_0^1 xde^{-x} = -A\left(xe^{-x}|_0^1 - \int_0^1 e^{-x}dx\right)$$

$$= -A(xe^{-x}|_0^1 + e^{-x}|_0^1) =$$

$$= -A(e^{-1} - 0 + e^{-1} - 1) = A(1-2e^{-1}) = \frac{1-2e^{-1}}{1-e^{-1}} = \frac{e-2}{e-1}$$

b) Variance

$$Var[X] = \sigma^2 = \int_0^1 (x-\mu)^2 f(x)dx = \int_0^1 Ae^{-x}(x-\mu)^2dx = \int_0^1 Ae^{-x}(x^2 - 2x\mu + \mu^2)dx =$$

$$= \int_0^1 Ae^{-x}x^2dx - \int_0^1 Ae^{-x}2x\mu dx + \int_0^1 Ae^{-x}\mu^2dx = A\int_0^1 e^{-x}x^2dx - 2\mu\int_0^1 Ae^{-x}xdx$$

$$+ \mu^2\int_0^1 Ae^{-x}dx = \int_0^1 Ae^{-x}x^2dx - 2\mu^2 + \mu^2 = \int_0^1 Ae^{-x}x^2dx - \mu^2$$

$$= E[X^2] - \mu^2$$

$$E[X^2] = \int_0^1 Ae^{-x}x^2dx = -A\int_0^1 x^2de^{-x} = -Ax^2e^{-x}|_0^1 + 2A\int_0^1 xe^{-x}dx =$$

$$= -Ae^{-1} + 2A(1-2e^{-1}) = A(2-5e^{-1})$$

$$\int_0^1 Ae^{-x}2x\mu dx = 2\mu\left(A\int_0^1 xe^{-x}dx\right) = 2\mu\cdot\mu = 2\mu^2$$

$$\int_0^1 Ae^{-x}\mu^2dx = \mu^2\int_0^1 Ae^{-x}dx = \mu^2\int_0^1 f(x)dx = \mu^2$$

$$\sigma^2 = A(2-5e^{-1}) - 2\mu^2 + \mu^2 =$$

$$= \frac{2-5e^{-1}}{1-e^{-1}} - \mu^2 = \frac{2-5e^{-1}}{1-e^{-1}} - \left(\frac{1-2e^{-1}}{1-e^{-1}}\right)^2 = \frac{1-3e^{-1}+4e^{-2}}{(1-e^{-1})^2}$$

c) $E[(2+3X)^2]$

$$E[(2+3X)^2] = E[4+12X+9X^2] = E[4] + E[12X] + E[9X^2] = 4 + 12E[X] + 9E[X^2]$$

$$= 4 + 12\mu + 9(Var[X] + \mu^2) = 4 + 12\frac{e-2}{e-1} + 9A(2-5e^{-1})$$

$$= 4 + 12\frac{e-2}{e-1} + 9\frac{e}{e-1}\frac{2e-5}{e} = \frac{4e-4+12e-24+18e-45}{e-1}$$

$$= \frac{34e-73}{e-1} = \frac{34-73e^{-1}}{1-e^{-1}}$$

because

$$E[(2+3X)^2] = \int_0^1 Ae^{-x}(3x+2)^2dx = \int_0^1 Ae^{-x}9x^2dx + \int_0^1 Ae^{-x}12xdx + \int_0^1 Ae^{-x}4dx =$$

$$= 9A(2-5e^{-1}) + 12A(1-2e^{-1}) + 4A(1-e^{-1}) = \frac{34-73e^{-1}}{1-e^{-1}}$$

Answer

a) $\mu = \frac{1-2e^{-1}}{1-e^{-1}}$

b) $\sigma^2 = \frac{1-3e^{-1}+4e^{-2}}{(1-e^{-1})^2}$

c) $E[(2+3x)^2] = \frac{34-73e^{-1}}{1-e^{-1}}$