Answer on Question #51422 – Math – integral Calculus

$\int 1\ /\ [(x)\verb|^|m\ +(x)\verb|^|n]\ dx\ =$? show process please ($m, n$ are integers).

Solution

Denote constant item by c.

1) $m = n$

$$\int \frac{dx}{2x^n} = \frac{1}{2}\int \frac{dx}{x^n} = \left[\begin{array}{l} \frac{1}{2}\ln|x| + c, n = 1 \\ \frac{x^{1-n}}{2(1-n)} + c, n \neq 1 \end{array}\right. .$$

2) $m \neq n$
Assume that $m > n$. Denote $k = m - n$.
So:

$$\int \frac{dx}{x^m + x^n} = \int \frac{dx}{x^{n+k} + x^n} = \int \frac{dx}{x^n(x^k+1)};$$

It follows from the fundamental theorem of algebra, that we can expand $x^k + 1$ into a product of linear and quadratic polynomials with real coefficients:

$$x^k + 1 = \left[\begin{array}{c} (x+1)(x^2 + a_1x + b_1)\dots(x^2 + a_px + b_p), 2 \nmid k \\ (x^2 + a_1x + b_1)\dots(x^2 + a_px + b_p), 2 \mid k \end{array}\right. ;$$

So we can expand fraction $\frac{1}{x^n(x^k+1)}$ into the following sum:

$$\frac{1}{x^n(x^k+1)} = \frac{c_1}{x} + \dots + \frac{c_n}{x^n} + \frac{\delta(k)c_{n+1}}{x+1} + \frac{d_1x + f_1}{x^2 + a_1x + b_1} + \dots + \frac{d_px + f_p}{x^2 + a_px + b_p},$$

where all coefficients c_i, d_i, f_i are real and $\delta(k) = \left[\begin{array}{l} 1, 2 \nmid k \\ 0, 2 \mid k \end{array}\right.;$

Hence:

$$\int \frac{dx}{x^n(x^k+1)} = c_1\int\frac{dx}{x} + \dots + c_n\int\frac{dx}{x^n} + \delta(k)c_{n+1}\int\frac{dx}{x+1} + \int\frac{d_1x + f_1}{x^2 + a_1x + b_1}dx +$$
$$+ \dots + \int\frac{d_px + f_p}{x^2 + a_px + b_p}dx;$$

The first n integrals are computed in 1).

$$\int\frac{dx}{x+1} = \ln|x+1| + c;$$

The last p integrals can by computed in the following way:

$$\int\frac{dx + f}{x^2 + ax + b}dx = \int\frac{dx + f}{\left(x + \frac{a}{2}\right)^2 + b - \frac{a^2}{4}}dx = \left[x + \frac{a}{2} = y\right] =$$
$$= \int\frac{d'y + f'}{y^2 + b - \frac{a^2}{4}}dy = \frac{d'}{2}\int\frac{d(y^2)}{y^2 + b - \frac{a^2}{4}} + f'\int\frac{dy}{y^2 + b - \frac{a^2}{4}} =$$
$$= \frac{d'}{2}\ln\left|y^2 + b - \frac{a^2}{4}\right| + c + f'\int\frac{dy}{y^2 + b - \frac{a^2}{4}} =$$

$$= \frac{d'}{2} \ln \left| (x + \frac{a}{2})^2 + b - \frac{a^2}{4} \right| + c + f' \int \frac{dy}{y^2 + b - \frac{a^2}{4}};$$

$$\int \frac{dy}{y^2 + b - \frac{a^2}{4}} = \begin{cases} \frac{1}{\sqrt{b - \frac{a^2}{4}}} \arctan \frac{y}{\sqrt{b - \frac{a^2}{4}}} + c, b - \frac{a^2}{4} > 0 \\ -\frac{1}{y} + c, b - \frac{a^2}{4} = 0 \\ \frac{1}{2\sqrt{b - \frac{a^2}{4}}} \ln \left| \frac{x - \sqrt{b - \frac{a^2}{4}}}{x + \sqrt{b - \frac{a^2}{4}}} \right|, b - \frac{a^2}{4} < 0 \end{cases}.$$