Answer on Question #51421 – Math – Integral Calculus:

$\int \frac{dx}{x^3+x^8}$ =? Show process please.

Solution

Denote $\phi = \frac{\sqrt{5}-1}{2}$. So:

$$\phi^2 = \frac{6-2\sqrt{5}}{4} = \frac{3-\sqrt{5}}{2} = 1-\phi;$$

We have that:

$$(x^2+\phi x+1)\left(x^2-\frac{1}{\phi}x+1\right) = x^4+\left(\phi-\frac{1}{\phi}\right)x^3+x^2+\left(\phi-\frac{1}{\phi}\right)x+1 =$$

$$= x^4+\frac{\phi^2-1}{\phi}x^3+x^2+\frac{\phi^2-1}{\phi}x+1 = x^4+\frac{-\phi}{\phi}x^3+x^2+\frac{-\phi}{\phi}x+1 =$$

$$= x^4-x^3+x^2-x+1.$$

So:

$$x^3+x^8 = x^3(x^5+1) = x^3(x+1)(x^4-x^3+x^2-x+1) =$$

$$= x^3(x+1)(x^2+\phi x+1)\left(x^2-\frac{1}{\phi}x+1\right);$$

Expand $\frac{1}{x^3+x^8}$ into a sum of fractions:

$$\frac{1}{x^3+x^8} = \frac{a}{x}+\frac{b}{x^2}+\frac{c}{x^3}+\frac{d}{x+1}+\frac{fx+g}{x^2+\phi x+1}+\frac{hx+k}{x^2-\frac{1}{\phi}x+1};$$

Now find a, b, c, d, f, g, h, k by the method of undetermined coefficients:

$$1 = ax^2(x^5+1)+bx(x^5+1)+c(x^5+1)+dx^3(x^4-x^3+x^2-x+1)+$$

$$+(fx+g)x^3(x+1)\left(x^2-\frac{1}{\phi}x+1\right)+(hx+k)x^3(x+1)(x^2+\phi x+1) \Rightarrow$$

$$\Rightarrow 1 = (a+d+f+h)x^7+\left(b-d+g+f-\frac{f}{\phi}+h+k+\phi h\right)x^6+$$

$$+\left(c+d+g-\frac{g}{\phi}-\frac{f}{\phi}+f+k+\phi k+\phi h+h\right)x^5+$$

$$+\left(-d+f+g-\frac{g}{\phi}+h+k+\phi k\right)x^4+(d+g+k)x^3+ax^2+bx+c \Rightarrow$$

$$\Rightarrow \begin{cases} a = b = 0, c = 1 \\ a + d + f + h = 0 \\ d + g + k = 0 \\ -d + f + g - \dfrac{g}{\phi} + h + k + \phi k = 0 \\ c + d + g - \dfrac{g}{\phi} - \dfrac{f}{\phi} + f + k + \phi k + \phi h + h = 0 \\ b - d + g + f - \dfrac{f}{\phi} + h + k + \phi h = 0 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} a = b = 0, c = 1 \\ d + f + h = 0 \\ d + g + k = 0 \\ -3d - \dfrac{g}{\phi} + \phi k = 0 \\ 1 - d - \dfrac{g}{\phi} - \dfrac{f}{\phi} + \phi k + \phi h = 0 \\ -3d - \dfrac{f}{\phi} + \phi h = 0 \end{cases} \Rightarrow \begin{cases} a = b = 0, c = 1 \\ d + f + h = 0 \\ g + k = f + h \\ -\dfrac{g}{\phi} + \phi k = 3d \\ 1 - d - \dfrac{g}{\phi} - \dfrac{f}{\phi} + \phi k + \phi h = 0 \\ \phi(h - k) = \dfrac{f - g}{\phi} \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} a = b = 0, c = 1 \\ d + f + h = 0 \\ g + k = f + h \\ -\dfrac{g}{\phi} + \phi k = 3d \\ 1 - d - \dfrac{g}{\phi} - \dfrac{f}{\phi} + \phi k + \phi h = 0 \\ (f - g)\left(\phi + \dfrac{1}{\phi}\right) = 0 \end{cases} \Rightarrow \begin{cases} a = b = 0, c = 1 \\ d + f + h = 0 \\ g + k = f + h \\ -\dfrac{g}{\phi} + \phi k = 3d \\ 1 - d - \dfrac{g}{\phi} - \dfrac{f}{\phi} + \phi k + \phi h = 0 \\ f = g \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} a = b = 0, c = 1 \\ d + f + h = 0 \\ h = k \\ -\dfrac{f}{\phi} + \phi h = 3d \\ 1 - d - \dfrac{2f}{\phi} + 2\phi h = 0 \\ f = g \end{cases} \Rightarrow \begin{cases} a = b = 0, c = 1 \\ d + f + h = 0 \\ h = k \\ \phi h - 3d = \dfrac{f}{\phi} \\ 1 + 5d = 0 \\ f = g \end{cases} \Rightarrow \begin{cases} a = b = 0, c = 1 \\ f + h = \dfrac{1}{5} \\ h = k \\ \phi h + \dfrac{3}{5} = \dfrac{f}{\phi} \\ d = -\dfrac{1}{5} \\ f = g \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} a = b = 0, c = 1 \\ h = \frac{1}{5} - f \\ h = k \\ \phi\left(\frac{1}{5} - f\right) + \frac{3}{5} = \frac{f}{\phi} \\ d = -\frac{1}{5} \\ f = g \end{cases} \Rightarrow \begin{cases} a = b = 0, c = 1 \\ h = \dfrac{\frac{1}{\phi} - 3}{5\left(\phi + \frac{1}{\phi}\right)} \\ k = \dfrac{\frac{1}{\phi} - 3}{5\left(\phi + \frac{1}{\phi}\right)} \\ f = \dfrac{\phi + 3}{5\left(\phi + \frac{1}{\phi}\right)} \\ d = -\frac{1}{5} \\ g = \dfrac{\phi + 3}{5\left(\phi + \frac{1}{\phi}\right)} \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} a = b = 0, c = 1 \\ d = -\frac{1}{5} \\ f = g = \frac{1}{5\phi} \\ h = k = -\frac{\phi}{5} \end{cases}$$

So:

$$\int \frac{dx}{x^3 + x^8} = \int \left(\frac{1}{x^3} - \frac{1}{5(x+1)} + \frac{1}{5\phi} \cdot \frac{x+1}{x^2 + \phi x + 1} - \frac{\phi}{5} \cdot \frac{x+1}{x^2 - \frac{1}{\phi}x + 1} \right) dx =$$

$$= \int \frac{dx}{x^3} - \frac{1}{5} \int \frac{dx}{x+1} + \frac{1}{5\phi} \int \frac{x+1}{x^2 + \phi x + 1} dx - \frac{\phi}{5} \int \frac{x+1}{x^2 - \frac{1}{\phi}x + 1} dx =$$

$$= -\frac{1}{2x^2} - \frac{1}{5}\ln|x+1| + \frac{1}{5\phi} \int \frac{x+1}{x^2 + \phi x + 1} dx - \frac{\phi}{5} \int \frac{x+1}{x^2 - \frac{1}{\phi}x + 1} dx;$$

Compute the rest of integrals:

$$\int \frac{x+1}{x^2 + \phi x + 1} dx = \int \frac{x+1}{\left(x + \frac{\phi}{2}\right)^2 + 1 - \frac{\phi^2}{4}} dx = \left[y = \frac{x + \frac{\phi}{2}}{\sqrt{1 - \frac{\phi^2}{4}}} \right] =$$

$$= \int \frac{\sqrt{1 - \frac{\phi^2}{4}}\, y + 1 - \frac{\phi}{2}}{\left(1 - \frac{\phi^2}{4}\right)(y^2 + 1)} \cdot \sqrt{1 - \frac{\phi^2}{4}}\, dy = \int \frac{y + \dfrac{1 - \frac{\phi}{2}}{\sqrt{1 - \frac{\phi^2}{4}}}}{y^2 + 1} dy =$$

$$= \int \frac{y}{y^2+1}dy + \frac{1-\frac{\phi}{2}}{\sqrt{1-\frac{\phi^2}{4}}}\int \frac{dy}{y^2+1} = \frac{1}{2}\int \frac{d(y^2)}{y^2+1} + \frac{1-\frac{\phi}{2}}{\sqrt{1-\frac{\phi^2}{4}}}\arctan y =$$

$$= \frac{1}{2}\ln(y^2+1) + \frac{1-\frac{\phi}{2}}{\sqrt{1-\frac{\phi^2}{4}}}\arctan y + c =$$

$$= \frac{1}{2}\ln\left(\left(\frac{x+\frac{\phi}{2}}{\sqrt{1-\frac{\phi^2}{4}}}\right)^2 + 1\right) + \frac{1-\frac{\phi}{2}}{\sqrt{1-\frac{\phi^2}{4}}}\arctan\frac{x+\frac{\phi}{2}}{\sqrt{1-\frac{\phi^2}{4}}} + c =$$

$$= \frac{1}{2}\ln\left(\frac{x^2+\phi x+1}{1-\frac{\phi^2}{4}}\right) + \frac{1-\frac{\phi}{2}}{\sqrt{1-\frac{\phi^2}{4}}}\arctan\frac{x+\frac{\phi}{2}}{\sqrt{1-\frac{\phi^2}{4}}} + c;$$

c is an arbitrary real constant.

$$\int \frac{x+1}{x^2-\frac{1}{\phi}x+1}dx = \int \frac{x+1}{\left(x-\frac{1}{2\phi}\right)^2+1-\frac{1}{4\phi^2}}dx = \left[y = \frac{x-\frac{1}{2\phi}}{\sqrt{1-\frac{1}{4\phi^2}}}\right] =$$

$$= \int \frac{\sqrt{1-\frac{1}{4\phi^2}}\,y + 1 + \frac{1}{2\phi}}{\left(1-\frac{1}{4\phi^2}\right)(y^2+1)} \cdot \sqrt{1-\frac{1}{4\phi^2}}\,dy =$$

$$= \int \frac{y + \frac{1+\frac{1}{2\phi}}{\sqrt{1-\frac{1}{4\phi^2}}}}{y^2+1}dy = \int \frac{y}{y^2+1}dy + \frac{1+\frac{1}{2\phi}}{\sqrt{1-\frac{1}{4\phi^2}}}\int \frac{dy}{y^2+1} =$$

$$= \frac{1}{2}\ln(y^2+1) + \frac{1+\frac{1}{2\phi}}{\sqrt{1-\frac{1}{4\phi^2}}}\arctan y + c =$$

$$= \frac{1}{2}\ln\left(\left(\frac{x-\frac{1}{2\phi}}{\sqrt{1-\frac{1}{4\phi^2}}}\right)^2 + 1\right) + \frac{1+\frac{1}{2\phi}}{\sqrt{1-\frac{1}{4\phi^2}}}\arctan\frac{x-\frac{1}{2\phi}}{\sqrt{1-\frac{1}{4\phi^2}}} + c =$$

$$= \frac{1}{2}\ln\left(\frac{x^2 - \frac{x}{\phi} + 1}{1 - \frac{1}{4\phi^2}}\right) + \frac{1 + \frac{1}{2\phi}}{\sqrt{1 - \frac{1}{4\phi^2}}}\arctan\frac{x - \frac{1}{2\phi}}{\sqrt{1 - \frac{1}{4\phi^2}}} + c;$$

c is an arbitrary real constant.

We conclude that

$$\int \frac{dx}{x^3 + x^8} = -\frac{1}{2x^2} - \frac{1}{5}\ln|x + 1| +$$

$$+\frac{1}{5\phi}\left(\frac{1}{2}\ln\left(\frac{x^2 + \phi x + 1}{1 - \frac{\phi^2}{4}}\right) + \frac{1 - \frac{\phi}{2}}{\sqrt{1 - \frac{\phi^2}{4}}}\arctan\frac{x + \frac{\phi}{2}}{\sqrt{1 - \frac{\phi^2}{4}}} + c\right) -$$

$$-\frac{\phi}{5}\left(\frac{1}{2}\ln\left(\frac{x^2 - \frac{x}{\phi} + 1}{1 - \frac{1}{4\phi^2}}\right) + \frac{1 + \frac{1}{2\phi}}{\sqrt{1 - \frac{1}{4\phi^2}}}\arctan\frac{x - \frac{1}{2\phi}}{\sqrt{1 - \frac{1}{4\phi^2}}} + c\right) =$$

$$= -\frac{1}{2x^2} - \frac{1}{5}\ln|x + 1| + \frac{1}{5\phi}\left(\frac{1}{2}\ln\left(\frac{x^2 + \phi x + 1}{1 - \frac{\phi^2}{4}}\right) + \sqrt{\frac{1 - \frac{\phi}{2}}{1 + \frac{\phi}{2}}}\arctan\frac{x + \frac{\phi}{2}}{\sqrt{1 - \frac{\phi^2}{4}}}\right) -$$

$$-\frac{\phi}{5}\left(\frac{1}{2}\ln\left(\frac{x^2 - \frac{x}{\phi} + 1}{1 - \frac{1}{4\phi^2}}\right) + \sqrt{\frac{1 + \frac{1}{2\phi}}{1 - \frac{1}{2\phi}}}\arctan\frac{x - \frac{1}{2\phi}}{\sqrt{1 - \frac{1}{4\phi^2}}}\right) + c =$$

$$= -\frac{1}{2x^2} - \frac{1}{5}\ln|x + 1| + \frac{1}{5\phi}\left(\frac{1}{2}\ln(x^2 + \phi x + 1) + \sqrt{\frac{1 - \frac{\phi}{2}}{1 + \frac{\phi}{2}}}\arctan\frac{x + \frac{\phi}{2}}{\sqrt{1 - \frac{\phi^2}{4}}}\right) -$$

$$-\frac{\phi}{5}\left(\frac{1}{2}\ln\left(x^2 - \frac{x}{\phi} + 1\right) + \sqrt{\frac{1 + \frac{1}{2\phi}}{1 - \frac{1}{2\phi}}}\arctan\frac{x - \frac{1}{2\phi}}{\sqrt{1 - \frac{1}{4\phi^2}}}\right) + c,$$

c is an arbitrary real constant.