Answer on Question #50266 – Math – Complex Analysis

Find the value(s) of constant B such that :
integral on curve for [(1)- (3z) + (2 B {z^4}) +(z^6) + (3 {z^7}) + (11 {z ^ 100}] /[z^5] dz =
integral on curve for [e ^ {Bz} + 2 z] / [z^2] dz
where C is the unit circle oriented counterclockwise

Solution

We use Cauchy`s integral formula for closed Jordan regions

$$\frac{1}{2\pi i}\oint_T \frac{f(z)}{z-z_0}dz = \begin{cases} f(z_0), & if\ z_0\ inside\ the\ closed\ contour \\ 0, & if\ z_0\ outside\ the\ closed\ contour \end{cases}$$

and its consequence (Cauchy`s differentiation formula)

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_T \frac{f(z)}{(z-z_0)^{n+1}}dz,$$

hence

$$\oint_T \frac{f(z)}{(z-z_0)^{n+1}}dz = \frac{2\pi i f^{(n)}(z_0)}{n!} \qquad (1)$$

In the given problem $z_0 = 0$ is inside the closed region, curve T is the unit circle.

In the first integral $\oint_{|z|=1} \frac{1-3z+2Bz^4+z^6+3z^7+11z^{100}}{z^5}dz$ choose

$$f(z) = 1 - 3z + 2Bz^4 + z^6 + 3z^7 + 11z^{100}, n = 4,$$

In the second integral $\oint_{|z|=1} \frac{e^{Bz}+2z}{z^2}dz$ take $g(z) = e^{Bz} + 2z, n = 1.$

To differentiate expressions, we need the following formula:

$$\left(z^k\right)^{(n)} = \begin{cases} \frac{k!}{(k-n)!}z^{k-n}, if\ k \geq n, \\ 0, \qquad if\ k < n \end{cases}$$

Next, compute

$$f^{(4)}(z) = (1 - 3z + 2Bz^4 + z^6 + 3z^7 + 11z^{100})^{(4)} = |linearity\ of\ the\ nth\ derivative| =$$
$$= (z^0)^{(4)} - 3(z^1)^{(4)} + 2B(z^4)^{(4)} + (z^6)^{(4)} + 3(z^7)^{(4)} + 11(z^{100})^{(4)} = 0 - 0 + 2B\frac{4!}{(4-4)!}z^{4-4} +$$
$$+\frac{6!}{(6-4)!}z^{6-4} + 3\frac{7!}{(7-4)!}z^{7-4} + 11\frac{100!}{(100-4)!}z^{100-4} = 2B \cdot 4! + \frac{6\cdot5\cdot4\cdot3\cdot2!}{2!}z^2 + 3\frac{7\cdot6\cdot5\cdot4\cdot3!}{3!}z^3 +$$
$$+11\frac{100\cdot99\cdot98\cdot97\cdot96!}{96!}z^{96},$$

$$f^{(4)}(0) = 2B \cdot 4! = 4 \cdot 3 \cdot 2 \cdot 2B = 48B,$$

$$g'(z) = (e^{Bz} + 2z)' = (e^{Bz})' + 2z' = (e^{Bz})' + 2 = Be^{Bz} + 2$$

$$g'(0) = B + 2$$

Using formula (1), rewrite equality

$$\oint_{|z|=1} \frac{1-3z+2Bz^4+z^6+3z^7+11z^{100}}{z^5} dz = \oint_{|z|=1} \frac{e^{Bz}+2z}{z^2} dz$$

as

$$\frac{2\pi i f^{(4)}(0)}{4!} = \frac{2\pi i g^{'}(0)}{1!},$$

which gives

$$\frac{f^{(4)}(0)}{4\cdot 3\cdot 2!} = \frac{g^{'}(0)}{1!},$$

so

$$f^{(4)}(0) = 4\cdot 3\cdot 2\cdot g^{'}(0),$$

which is equivalent to

$$48B = 24(B+2),$$

i.e.

$$48B - 24B = 48,$$

hence

$$24B = 48$$

and finally obtain

$$B = 2\ .$$

Answer: $B = 2$