Answer on the Question #50262 – Math – Complex Analysis

Given:

$$f(z)=e^{z^2+2z-3} \qquad z_0=-1$$

Find the taylor series of f(z)= e^ { z^2 + 2z -3} about z=-1 along with its convergent neighborhood

Solution:

Method 1 (application of expansion for exponential function and product of series):

$$f(z)=\frac{e^{z^2}\cdot e^{2z}}{e^3}=\frac{1}{e^3}\cdot e^{z^2-1+1}\cdot e^{2z+2-2}=\frac{1}{e^3}\cdot e\cdot e^{z^2-1}\cdot e^{-2}\cdot e^{2z+2}=\frac{1}{e^4}\cdot\sum_{n=0}^{\infty}\frac{(z^2-1)^n}{n!}\cdot\sum_{n=0}^{\infty}\frac{(2z+2)^n}{n!}=$$

$$=\frac{1}{e^4}\cdot\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{(z^2-1)^k}{k!}\cdot\frac{(2z+2)^{n-k}}{(n-k)!}\right)$$

Method 2 (application of Taylor`s formula):

$$f(z)=e^{z^2+2z-3}=e^{(z+1)^2-4}=\frac{1}{e^4}\cdot e^{(z+1)^2}=$$

$$=\frac{1}{e^4}\left(g(-1)+g'(-1)(z+1)+\frac{g''(-1)}{2!}(z+1)^2+...+\frac{g^{(n)}(-1)}{n!}(z+1)^n+...\right),$$

where $g(z)=e^{(z+1)^2}$, $g(-1)=e^0=1$,

$$g'(z)=e^{(z+1)^2}(2z+2),\ g'(-1)=e^0(-2+2)=0,$$

$$g''(z)=\left(e^{z^2+2z+1}\right)'(2z+2)+e^{z^2+2z+1}(2z+2)'=e^{z^2+2z+1}(2z+2)^2+2e^{z^2+2z+1},$$

$$g''(-1)=e^{z^2+2z+1}(-2+2)^2+2e^{z^2+2z+1}=2,$$

$$g'''(z)=\left(e^{z^2+2z+1}\right)'(4z^2+8z+6)+e^{z^2+2z+1}(4z^2+8z+6)'=$$

$$=e^{z^2+2z+1}(4z^2+8z+6)(2z+2)+e^{z^2+2z+1}(8z+8)$$

$$g'''(-1)=e^0(4-8+6)(-2+2)+e^0(-8+8)=0$$

and so on.

$$f(z)=e^{z^2+2z-3}=\frac{1}{e^4}\left(1+(z+1)^2+...\right)$$

Method 3 (application of expansion for exponential function and multinomial formula):

Since $e^z=\sum_{n=0}^{\infty}\frac{z^n}{n!}$, then

$$f(z)=e^{z^2+2z-3}=e^{(z+1)^2-4}=\frac{1}{e^4}\cdot e^{(z+1)^2}=$$

$$\frac{1}{e^4}\sum_{n=0}^{\infty}\frac{\left((z+1)^2\right)^n}{n!}=\frac{1}{e^4}\sum_{n=0}^{\infty}\frac{\left(z^2+2z+1\right)^n}{n!}=$$

$$\sum_{n=0}^{\infty}\left(\frac{\sum\limits_{k+l+m=n,k\geq 0,l\geq 0,m\geq 0}\frac{n!}{k!l!m!}2^l z^{2k} z^l}{n!}\right).$$

Answer: $f(z)=e^{z^2+2z-3}=\frac{1}{e^4}\left(1+(z+1)^2+\ldots\right)$