Answer on Question #50258 – Math – Complex Analysis

Use Cauchy Integral Formula to evaluate

Integral on Curve [e ^ (z+1)] / [{(z-i) (z^2 + (i-1)z-i) }^3]

Note : 1)) please the figure is close curve I cannot paint it by writing but these points (3i,i,-i,-2i,-1) inside the figure if you need it when find the singularity inside the curve
2)) Also I need all singularity (then sure only if inside curve i need all one with the working of cauchy integral , then plus it to find the total integral

Solution

$\oint_C \frac{e^{z+1}}{(z-i)(z^2+(i-1)z-i)^3} dz,$

where C — closed curve.

Let us find singularities:

Numerator hasn't finite zeros. Denominator certainly has some, so let's find them.

Consider

$$z - i = 0$$

$$z_1 = i$$

Consider

$$(z^2 + (i-1)z - i)^3 = 0$$

$$z^2 + (i-1)z - i = 0$$

Due to the Vieta's formulas:

$$z_2 + z_3 = -\frac{i-1}{1} = 1 - i$$

$$z_2 z_3 = -\frac{i}{1} = -i$$

It's obvious now that

$$z_2 = 1$$

$$z_3 = -i$$

As we said, numerator hasn't finite zeros, also $z_1 \neq z_2 \neq z_3$, thus, zeros 1 and $-i$ of denominator are nothing else, but poles of order 3, zero I of denominator is simple pole (pole of order 1)

$$\frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}} dz = f^{(n)}(a)$$

is called Cauchy's integral formula,

hence

$$\oint_C \frac{f(z)}{(z-a)^{n+1}}dz = \frac{2\pi i}{n!}f^{(n)}(a)$$

If region enclosed with a curve contains more than one singularity, we split up area/curve into parts, which contain each only one singularity. Let the singularity 1 is not included in the region.

Trigonometric form of complex number (angle between $-\pi$ and π):

$$i+i=2i=2\left(cos\left(\frac{\pi}{2}\right)+isin\left(\frac{\pi}{2}\right)\right),$$

$$i-1=\sqrt{2}\left(-\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)=\sqrt{2}\left(cos\left(\frac{3\pi}{4}\right)+isin\left(\frac{3\pi}{4}\right)\right)$$

$$(i-1)^3=2^{3/2}\left(cos\left(\frac{3\pi\cdot 3}{4}\right)+isin\left(\frac{3\pi\cdot 3}{4}\right)\right)=2\sqrt{2}\left(cos\left(\frac{9\pi}{4}\right)+isin\left(\frac{9\pi}{4}\right)\right)$$
$$=2\sqrt{2}\left(cos\left(\frac{\pi}{4}\right)+isin\left(\frac{\pi}{4}\right)\right)=2\sqrt{2}\frac{1+i}{\sqrt{2}}=2+2i$$

By Euler formula, $e^{i+1}=e\cdot e^{i}=e(\cos(1)+isin(1))$

For $z=i$ we have $f(z)=\frac{e^{z+1}}{(z+i)^3(z-1)^3}$, $n=1, a=i$,

$$\oint_{C_1}\frac{\frac{e^{z+1}}{(z+i)^3(z-1)^3}}{z-i}dz=2\pi i f(i)=2\pi i\frac{e^{i+1}}{(i+i)^3(i-1)^3}$$
$$=2\pi i\frac{e(\cos(1)+isin(1))}{2^3\left(cos\left(\frac{3\pi}{2}\right)+isin\left(\frac{3\pi}{2}\right)\right)2^{\frac{3}{2}}\left(cos\left(\frac{3\pi}{4}\right)+isin\left(\frac{3\pi}{4}\right)\right)}$$
$$=2\pi i\frac{e(\cos(1)+isin(1))\left(cos\left(-\frac{3\pi}{2}-\frac{3\pi}{4}\right)+isin\left(-\frac{3\pi}{2}-\frac{3\pi}{4}\right)\right)}{2^3 2^{\frac{3}{2}}}$$
$$=2\pi i\frac{e(\cos(1)+isin(1))\left(cos\left(-\frac{9\pi}{4}\right)+isin\left(-\frac{9\pi}{4}\right)\right)}{2^3 2^{\frac{3}{2}}}$$
$$=2\pi i\frac{e(\cos(1)+isin(1))\left(cos\left(-\frac{\pi}{4}\right)+isin\left(-\frac{\pi}{4}\right)\right)}{2^3 2^{\frac{3}{2}}}$$
$$=2\pi i\frac{e(\cos(1)+isin(1))(1-i)}{2^3 2^{\frac{3}{2}}2^{\frac{1}{2}}}=\pi e^{1+i}\frac{1+i}{16}$$

For $z=-i$ we have $f(z)=\frac{e^{z+1}}{(z-i)(z-1)^3}$, $n=2, a=-i$,

$\oint_{C_2} \frac{\frac{e^{z+1}}{(z-i)(z-1)^3}}{(z+i)^3} dz = \frac{2\pi i}{2} f''(-i) = \pi i f''(-i) = \pi e^{1-i} \frac{19i-27}{16}.$

Method 1.

$$f'(z) = \left(\frac{e^{z+1}}{(z-i)(z-1)^3}\right)' = \frac{(e^{z+1})'(z-i)(z-1)^3 - e^{z+1}((z-i)(z-1)^3)'}{(z-i)^2(z-1)^6}$$
$$= e^{z+1} \frac{(z-i)(z-1)^3 - 3(z-1)^2(z-i) - (z-1)^3}{(z-i)^2(z-1)^6}$$
$$= e^{z+1} \frac{(z-i)(z-1) - 3(z-i) - (z-1)}{(z-i)^2(z-1)^4}$$
$$= e^{z+1} \frac{z^2 - z - iz + i - 3z + 3i - z + 1}{(z-i)^2(z-1)^4} = e^{z+1} \frac{z^2 - (5+i)z + 4i + 1}{(z-i)^2(z-1)^4}$$

$$f''(z) = \left(e^{z+1} \frac{z^2 - (5+i)z + 4i + 1}{(z-i)^2(z-1)^4}\right)'$$
$$= (e^{z+1})' \frac{z^2 - (5+i)z + 4i + 1}{(z-i)^2(z-1)^4} + e^{z+1} \left(\frac{z^2 - (5+i)z + 4i + 1}{(z-i)^2(z-1)^4}\right)'$$
$$= e^{z+1} \left(\frac{z^2 - (5+i)z + 4i + 1}{(z-i)^2(z-1)^4} + \frac{(2z-5-i)(z-i)^2(z-1)^4 - (z^2-(5+i)z+4i+1)[2(z-i)(z-1)^4 + 4(z-i)^2(z-1)^3]}{(z-i)^4(z-1)^8}\right)$$
$$= e^{z+1} \left(\frac{z^2 - (5+i)z + 4i + 1}{(z-i)^2(z-1)^4} + \frac{(2z-5-i)(z-i)(z-1) - (z^2-(5+i)z+4i+1)[2(z-1) + 4(z-i)]}{(z-i)^4(z-1)^8}\right)$$

Method 2.

$$\frac{1}{(z-i)(z-1)^3} = \frac{A}{z-i} + \frac{B}{(z-1)^3} + \frac{C}{(z-1)^2} + \frac{D}{z-1}$$
$$= \frac{A(z-1)^3 + B(z-i) + C(z-1)(z-i) + D(z-i)(z-1)^2}{(z-i)(z-1)^3}$$

Equate the left-hand and the right-hand sides of the previous equalities.

If $z = 1$ then $B(1-i) = 1$, hence $B = \frac{1}{1-i} = \frac{1+i}{(1-i)(1+i)} = \frac{1+i}{1-i^2} = \frac{1+i}{1+1} = \frac{1+i}{2}$

If $z = i$ then $A(i-1)^3 = 1$, hence

$A = \frac{1}{(i-1)^3} = \frac{(i+1)^3}{(i^2-1)^3} = \frac{i^3+3i^2+3i+1}{(-2)^3} = \frac{i^2 \cdot i + 3\cdot(-1) + 3i + 1}{-8} = \frac{3i - i + 1 - 3}{-8} = \frac{2i-2}{-8} = \frac{1-i}{4}$

Consider $\frac{1}{(z-i)(z-1)^3} - \frac{A}{z-i} - \frac{B}{(z-1)^3} = \frac{C}{(z-1)^2} + \frac{D}{z-1}$,

$$\frac{1}{(z-i)(z-1)^3} - \frac{1-i}{4(z-i)} - \frac{1+i}{2(z-1)^3} = \frac{C}{(z-1)^2} + \frac{D}{z-1}$$

$\frac{4 - (1-i)(z-1)^3 - 2(1+i)(z-i)}{4(z-i)(z-1)^3} = \frac{4C(z-i)(z-1) + 4D(z-i)(z-1)^2}{4(z-i)(z-1)^3}$, hence

$4-(1-i)(z^3-3z^2+3z-1)-2(1+i)(z-i)=4C(z^2-(1+i)z+i)+4D(z-i)(z^2-2z+1)$,

$(i-1)z^3+(3-3i)z^2+\big((3i-3)-2(1+i)\big)z+1-i+4+2i+2i^2=4Cz^2-4C(1+i)z+4Ci+4D(z^3-2z^2+z-iz^2+2iz-i)$,

$(i-1)z^3+(3-3i)z^2+(i-5)z+3+i=4Dz^3+(4C-8D-4Di)z^2+(-4C(1+i)+4D+8Di)z+4Ci-4Di$,

hence $4D=i-1$, $4C-8D-4Di=3-3i$, $-4C(1+i)+4D+8Di=i-5$, $4Ci-4Di=3+i$

Solve for $D=\frac{i-1}{4}$, substitute into other expressions: $4C-2i+2-i(i-1)=3-3i$, $-4C(1+i)+i-1+i(2i-2)=i-5$, $4Ci-i(i-1)=i+3$.

Simplify these expressions: $4C-i+3=3-3i$, $-4C(1+i)-i-3=i-5$, $4Ci+i+1=i+3$, so $C=-\frac{i}{2}$, check $2i(1+i)-i-3=i-5$, $-2i^2+i+1=i+3$, which also hold true.

Thus,

$A=\frac{1-i}{4}, B=\frac{1+i}{2}, C=-\frac{i}{2}, D=\frac{i-1}{4}$

$$\frac{1}{(z-i)(z-1)^3}=\frac{A}{z-i}+\frac{B}{(z-1)^3}+\frac{C}{(z-1)^2}+\frac{D}{z-1}=$$
$$=\frac{1+i}{2(z-1)^3}-\frac{i}{2(z-1)^2}+\frac{i-1}{4(z-1)}+\frac{1-i}{4(z-i)}$$

Trigonometric form of complex number (angle between $-\pi$ and π):

$-i-1=\sqrt{2}\left(-\frac{1}{\sqrt{2}}-i\frac{1}{\sqrt{2}}\right)=\sqrt{2}\left(cos\left(-\frac{3\pi}{4}\right)+isin\left(-\frac{3\pi}{4}\right)\right)$,

hence

$$(-i-1)^{-5}=2^{-\frac{5}{2}}\left(cos\left(\frac{3\pi\cdot 5}{4}\right)+isin\left(\frac{3\pi\cdot 5}{4}\right)\right)=\frac{1}{4\sqrt{2}}\left(cos\left(\frac{15\pi}{4}\right)+isin\left(\frac{15\pi}{4}\right)\right)=$$

$=\frac{1}{4\sqrt{2}}\left(cos\left(-\frac{\pi}{4}\right)+isin\left(-\frac{\pi}{4}\right)\right)=\frac{1}{4\sqrt{2}}\left(\frac{1}{\sqrt{2}}-i\frac{1}{\sqrt{2}}\right)=\frac{1-i}{16}$,

$(-i-1)^{-4}=2^{-\frac{4}{2}}\big(cos(3\pi)+isin(3\pi)\big)=\frac{1}{4}\big(cos(\pi)+isin(\pi)\big)=-\frac{1}{4}$,

$(-i-1)^{-3}=2^{-\frac{3}{2}}\left(cos\left(\frac{3\pi\cdot 3}{4}\right)+isin\left(\frac{3\pi\cdot 3}{4}\right)\right)=\frac{1}{2\sqrt{2}}\left(cos\left(\frac{9\pi}{4}\right)+isin\left(\frac{9\pi}{4}\right)\right)=\frac{1}{2\sqrt{2}}\left(cos\left(\frac{\pi}{4}\right)+isin\left(\frac{\pi}{4}\right)\right)=\frac{1}{2\sqrt{2}}\left(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)=\frac{i+1}{4}$,

$$(-i-1)^{-2}=2^{-\frac{2}{2}}\left(cos\left(\frac{3\pi\cdot 2}{4}\right)+isin\left(\frac{3\pi\cdot 2}{4}\right)\right)=\frac{1}{2}\left(cos\left(\frac{6\pi}{4}\right)+isin\left(\frac{6\pi}{4}\right)\right)=$$

$= \frac{1}{2}\left(cos\left(\frac{3\pi}{2}\right) + isin\left(\frac{3\pi}{2}\right)\right) = \frac{-i}{2}$,

$(-i-1)^{-1} = 2^{-\frac{1}{2}}\left(cos\left(\frac{3\pi}{4}\right) + isin\left(\frac{3\pi}{4}\right)\right) = \frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \frac{i-1}{2}$.

Applying formulas $\left((ax+b)^{\beta}\right)^{(n)} = a^n\beta(\beta-1)\dots(\beta-n+1)(ax+b)^{\beta-n}$, obtain

$$((z-1)^{-3})^{(2)} = -3(-3-1)(z-1)^{-3-2} = \frac{12}{(z-1)^5}$$

$$((z-1)^{-3})^{(1)} = -3(z-1)^{-3-1} = -\frac{3}{(z-1)^4}$$

$$((z-1)^{-2})^{(2)} = -2(-2-1)(z-1)^{-2-2} = \frac{6}{(z-1)^4}$$

$$((z-1)^{-2})^{(1)} = -2(z-1)^{-2-1} = -\frac{2}{(z-1)^3}$$

$$((z-1)^{-1})^{(2)} = -1(-1-1)(z-1)^{-1-2} = \frac{2}{(z-1)^3}$$

$$((z-1)^{-1})^{(1)} = -(z-1)^{-1-1} = -\frac{1}{(z-1)^2}$$

$$((z-i)^{-1})^{(2)} = -1(-1-1)(z-i)^{-1-2} = \frac{2}{(z-i)^3}$$

$$((z-i)^{-1})^{(1)} = -(z-i)^{-1-1} = -\frac{1}{(z-i)^2}$$

Besides, $(e^{z+1})' = (e^{z+1})'' = e^{z+1}$

Using Leibnitz formula, rewrite

$$\left(\frac{e^{z+1}}{(z-i)(z-1)^3}\right)'' = \frac{1+i}{2}\left(\frac{e^{z+1}}{(z-1)^3}\right)'' - \frac{i}{2}\left(\frac{e^{z+1}}{(z-1)^2}\right)'' + \frac{i-1}{4}\left(\frac{e^{z+1}}{z-1}\right)'' + \frac{1-i}{4}\left(\frac{e^{z+1}}{z-i}\right)'' =$$

$$= \frac{1+i}{2}(e^{z+1}(z-1)^{-3})'' - \frac{i}{2}(e^{z+1}(z-1)^{-2})'' + \frac{i-1}{4}(e^{z+1}(z-1)^{-1})''$$
$$+ \frac{1-i}{4}(e^{z+1}(z-i)^{-1})'' =$$

$$= \frac{1+i}{2}\left[(e^{z+1})^{(2)}(z-1)^{-3} + 2(e^{z+1})^{(1)}((z-1)^{-3})^{(1)} + e^{z+1}((z-1)^{-3})^{(2)}\right]$$

$$-\frac{i}{2}\left[(e^{z+1})^{(2)}(z-1)^{-2} + 2(e^{z+1})^{(1)}((z-1)^{-2})^{(1)} + e^{z+1}((z-1)^{-2})^{(2)}\right]$$

$$+\frac{i-1}{4}\left[(e^{z+1})^{(2)}(z-1)^{-1} + 2(e^{z+1})^{(1)}((z-1)^{-1})^{(1)} + e^{z+1}((z-1)^{-1})^{(2)}\right] +$$

$$\frac{1-i}{4}\left[(e^{z+1})^{(2)}(z-i)^{-1} + 2(e^{z+1})^{(1)}((z-i)^{-1})^{(1)} + e^{z+1}((z-i)^{-1})^{(2)}\right] =$$

$=\frac{1+i}{2}e^{z+1}\left[(z-1)^{-3} + 2((z-1)^{-3})^{(1)} + ((z-1)^{-3})^{(2)}\right]$

$$-\frac{i}{2}e^{z+1}\left[(z-1)^{-2}+2((z-1)^{-2})^{(1)}+((z-1)^{-2})^{(2)}\right]$$

$$+\frac{i-1}{4}e^{z+1}\left[(z-1)^{-1}+2((z-1)^{-1})^{(1)}+((z-1)^{-1})^{(2)}\right]+$$

$$+\frac{1-i}{4}e^{z+1}\left[(z-i)^{-1}+2((z-i)^{-1})^{(1)}+((z-i)^{-1})^{(2)}\right]=$$

$$=\frac{1+i}{2}e^{z+1}\left[\frac{1}{(z-1)^3}-\frac{6}{(z-1)^4}+\frac{12}{(z-1)^5}\right]$$

$$-\frac{i}{2}e^{z+1}\left[\frac{1}{(z-1)^2}-\frac{4}{(z-1)^3}+\frac{6}{(z-1)^4}\right]$$

$$+\frac{i-1}{4}e^{z+1}\left[\frac{1}{z-1}-\frac{2}{(z-1)^2}+\frac{2}{(z-1)^3}\right]+$$

$$+\frac{1-i}{4}e^{z+1}\left[\frac{1}{z-i}-\frac{2}{(z-i)^2}+\frac{2}{(z-i)^3}\right]=$$

$$=\frac{1+i}{2}e^{z+1}\left[\frac{z^2-2z+1-6z+6+12}{(z-1)^5}\right]$$

$$-\frac{i}{2}e^{z+1}\left[\frac{z^2-2z+1-4z+4+6}{(z-1)^4}\right]$$

$$+\frac{i-1}{4}e^{z+1}\left[\frac{z^2-2z+1-2z+2+2}{(z-1)^3}\right]+$$

$$+\frac{1-i}{4}e^{z+1}\left[\frac{z^2-2iz-1-2z+2i+2}{z-i}\right]=$$

$$=e^{z+1}\left(\frac{1+i}{2}\left[\frac{z^2-8z+19}{(z-1)^5}\right]-\frac{i}{2}\left[\frac{z^2-6z+11}{(z-1)^4}\right]+\frac{i-1}{4}\left[\frac{z^2-4z+5}{(z-1)^3}\right]+\frac{1-i}{4}\left[\frac{z^2-(2i+2)z+2i+1}{(z-i)^3}\right]\right),$$

$$\left(\frac{e^{z+1}}{(z-i)(z-1)^3}\right)''\Big|_{z=-i}$$
$$=e^{i+1}\left(\frac{1+i}{2}\left[\frac{i^2-8i+19}{(i-1)^5}\right]-\frac{i}{2}\left[\frac{i^2-6i+11}{(i-1)^4}\right]+\frac{i-1}{4}\left[\frac{i^2-4i+5}{(i-1)^3}\right]\right.$$
$$\left.+\frac{1-i}{4}\left[\frac{i^2-(2i+2)i+2i+1}{(z-i)^3}\right]\right)$$

If region enclosed with a curve contains more than one singularity, we split up area/curve into parts, which contain each only one singularity. Let the singularity 1 is not included in the region.

Thus,

$$\oint_C \frac{e^{z+1}}{(z-i)(z^2+(i-1)z-i)^3}dz=$$

$$\oint_C \frac{e^{z+1}}{(z-i)(z+i)^3(z-1)^3}dz =$$

$$\oint_{C_1} \frac{\frac{e^{z+1}}{(z+i)^3(z-1)^3}}{z-i}dz + \oint_{C_2} \frac{\frac{e^{z+1}}{(z-i)(z-1)^3}}{(z+i)^3}dz =$$

$$= 2\pi i \frac{e^{z+1}}{(z+i)^3(z-1)^3}\Big|_{z=i} + \pi i \left(\frac{e^{z+1}}{(z-i)(z-1)^3}\right)''\Big|_{z=-i} =$$

$$= \pi e^{1+i}\frac{1+i}{16} + \pi e^{1-i}\frac{19i-27}{16}$$

The region does not contain $z = 1$.