Answer on Question #50254 – Math – Complex Analysis

1. Use Residue theorem to compute
Principle value
integral from (- infinity to infinity)
[3x] / [(x^2 + 1) ^2 (x+2)] dx

Solution.

$$I = \int_{-\infty}^{+\infty} \frac{3x\,dx}{\left(x^2+1\right)^2(x+2)} = \int_{C^+} f(z)dz - \int_{z=-2} f(z)dz, \quad \text{where } f(z) = \frac{3z}{\left(z^2+1\right)^2(z+2)}.$$

The isolated singular points of the function are $z=-2$ (a single pole) and $z=\pm i$ (poles of order 2). Only $z=i$ is located above X-axis. The residue at this point: $\underset{z=i}{res}\, f = \lim_{z\to i}\frac{d}{dz} f\cdot(z-i)^2 =$

$$= \lim_{z\to i}\frac{d}{dz}\frac{3z}{(z+i)^2(z+2)} = 3\lim_{z\to i}\frac{1\cdot(z+i)^2(z+2) - z\cdot\left[2(z+i)(z+2)+(z+i)^2\right]}{(z+i)^4(z+2)^2} =$$

$$= \lim_{z\to i}\frac{6\left(i-z^2-z\right)}{(z+i)^3(z+2)^2} = \frac{3i}{4(i+2)^2}.$$

$$\int_{z=-2} f(z)dz = \begin{vmatrix} z=-2+re^{i\varphi} \\ dz = re^{i\varphi}\, i\,d\varphi \end{vmatrix} = \lim_{r\to+0}\int_{\pi}^{0}\frac{3\left(-2+re^{i\varphi}\right)re^{i\varphi}\, i\,d\varphi}{\left(\left\{-2+re^{i\varphi}\right\}^2+1\right)^2 re^{i\varphi}} = \int_{\pi}^{0}\frac{3(-2)i\,d\varphi}{\left(2^2+1\right)^2} = -\frac{6i}{25}\int_{\pi}^{0} d\varphi = \frac{6i\pi}{25}.$$

According to the residue theorem, $I = 2\pi i\cdot \underset{z=i}{res}\, f - \frac{6i\pi}{25} = 2\pi i\cdot\frac{3i}{4(i+2)^2} - \frac{6i\pi}{25} = -\frac{9\pi}{50}.$

Answer: $-\frac{9\pi}{50}$.