Answer on Question#50048 - <Math> - <Complex Analysis>

Let $f(z)=\frac{1+\cos z}{(z-1)^2-1}\sin^2 z$

1) find zeroes of f

Solution. Let's solve equation:

$f(z)=0;$

$$\frac{1+\cos z}{(z-1)^2-1}\sin^2 z=0 \Rightarrow \left[\begin{array}{l}1+\cos z=0\\ \sin^2 z=0\end{array}\right. \Leftrightarrow \left[\begin{array}{l}z=\pi+2\pi k, k\in Z\\ z=\pi k, k\in Z\end{array}\right.,\ \text{where } (z-1)^2-1\neq 0 \Leftrightarrow z_1\neq 0, z_2\neq 2$$

Answer: $z=\pi k, k\in Z\setminus\{0\}$ zeroes of f

2) classify the zero $z=0$, $z=\pi i$, $z=2\pi i$

Solution. First of all, we must note that

$f(\pi i)=\frac{1+\cos\pi i}{(\pi i-1)^2-1}\sin^2\pi i\neq 0$ and $f(2\pi i)=\frac{1+\cos 2\pi i}{(2\pi i-1)^2-1}\sin^2 2\pi i\neq 0$. So, this mean that in the task was a mistake. We assume that there was $z=0$, $z=\pi$, $z=2\pi$

Let's consider $z=0$:

Since, f didn't determined at $z=0$, but holomorphic in neighborhood around $z=0$, excluding $z=0$, then $z=0$ is the singular point.

Since, $\lim\limits_{z\to 0} f(z)=\lim\limits_{z\to 0}\frac{1+\cos z}{(z-1)^2-1}\sin^2 z=\lim\limits_{z\to 0}\frac{1+\cos z}{(z-2)}\frac{\sin z}{z}\sin z=0$, then $z=0$ is a removable singularity.

Let's consider $z=\pi$:

$f(z)=\frac{1+\cos z}{(z-1)^2-1}\sin^2 z=(z-\pi)^4\frac{1+\cos z}{z(z-2)}\frac{\sin^2 z}{(z-\pi)^4}=(z-\pi)^4 h(z)$, where $h(z)=\frac{1+\cos z}{z(z-2)}\frac{\sin^2 z}{(z-\pi)^4}$

Since, $\lim\limits_{z\to\pi}(z-\pi)h(z)=\lim\limits_{z\to\pi}\frac{1+\cos z}{z(z-2)}\frac{\sin^2 z}{(z-\pi)^3}=\lim\limits_{z\to\pi}\frac{1}{z(z-2)}\frac{\sin^2 z}{z-\pi}\frac{1+\cos z}{(z-\pi)^2}=\frac{0}{2\pi(\pi-2)}=0$,

but $\lim\limits_{z\to\pi}h(z)=\lim\limits_{z\to\pi}\frac{1+\cos z}{z(z-2)}\frac{\sin^2 z}{(z-\pi)^4}=\lim\limits_{z\to\pi}\frac{1}{z(z-2)}\frac{\sin^2 z}{(z-\pi)^2}\frac{1+\cos z}{(z-\pi)^2}=\frac{1}{2\pi(\pi-2)}\neq 0$, then $z=\pi$ is 4 order zero of f.

Let's consider $z=2\pi$:

$f(z)=\frac{1+\cos z}{(z-1)^2-1}\sin^2 z=(z-2\pi)^2\frac{1+\cos z}{z(z-2)}\frac{\sin^2 z}{(z-2\pi)^2}=(z-2\pi)^2 h(z)$, where

$h(z)=\frac{1+\cos z}{z(z-2)}\frac{\sin^2 z}{(z-2\pi)^2}$

Since, $\lim\limits_{z\to 2\pi}(z-2\pi)h(z)=\lim\limits_{z\to 2\pi}\frac{1+\cos z}{z(z-2)}\frac{\sin^2 z}{z-2\pi}=\frac{2}{2\pi(2\pi-2)}0=0$,

but $\lim\limits_{z\to 2\pi}h(z)=\lim\limits_{z\to 2\pi}\frac{1+\cos z}{z(z-2)}\frac{\sin^2 z}{(z-\pi)^2}=\lim\limits_{z\to\pi}\frac{1+\cos z}{z(z-2)}\frac{\sin^2 z}{(z-2\pi)^2}=\frac{2}{2\pi(2\pi-2)}\neq 0$, then $z=2\pi$ is 2 order zero of f.

Answer: $z=0$ is a removable singularity, $z=\pi$ is 4 order zero of f, $z=2\pi$ is 2 order zero of f.