Answer on Question# #47877 – Mathematics – Differential Calculus | Equations

Question:

Find the solution of the equation

$$\mathrm{div}(\mathrm{grad}\ z)(x, y) = e^{-x}\cos y, \tag{1}$$

which tends to zero as x tends to infinity.

Solution:

Let us rewrite equation (1) by means of the Laplace operator:

$$\mathrm{div}(\mathrm{grad}\ z)(x, y) = \left(\nabla \cdot \nabla z(x, y)\right) = \nabla^2 z(x, y) = \Delta z(x, y) = \frac{\partial^2 z(x,y)}{\partial x^2} + \frac{\partial^2 z(x,y)}{\partial y^2} = e^{-x}\cos y. \tag{1a}$$

Consider the right side of this equation. As the function $f(x, y) = e^{-x}\cos y$ is a periodic in *y*-direction, then we can search the solution in the form

$$z(x, y) = f(x)\cos y, \tag{2}$$

Substituting (2) in the equation (1a), we have

$$\frac{\partial^2 z(x, y)}{\partial x^2} = f''\cos y, \quad \frac{\partial^2 z(x, y)}{\partial y^2} = -f\cos y,$$

$$f''\cos y - f\cos y = e^{-x}\cos y,$$

$$f'' - f = e^{-x}, \tag{3}$$

Therefore we obtain the second-order linear ordinary differential equation. The general solution of this equation is

$$f(x) = C_1 e^{x} + C_1 e^{-x} - x\frac{e^{-x}}{2}, \tag{4}$$

By definition f(y)=*cosy* is the bounded function. So, from the problem condition it follows that

$$z(x, y)_{x\to\infty} \to 0 \Rightarrow f(x)_{x\to\infty} \to 0. \tag{5}$$

Now, using (5) and (4) we receive $\left(C_1 e^{x} + C_1 e^{-x} - x\frac{e^{-x}}{2}\right)_{x\to\infty} \to 0$

As we see, the limit condition holds, if $C_1 = 0$, $C_2 = 1$ (note that $\lim_{x\to\infty}\left(e^{-x} - x\frac{e^{-x}}{2}\right) = 0$). Thus, the solution of equation (1) is

$$z(x, y) = \left(e^{-x} - x\frac{e^{-x}}{2}\right)\cos y. \tag{6}$$

Answer: $z(x, y) = \left(e^{-x} - x\frac{e^{-x}}{2}\right)\cos y$.