Answer on Question #47595 – Math - Matrix | Tensor Analysis

1. Use technology to solve the appropriate matrix equation to find the quadratic equation that is the best fit to the points (-2, 25), (3,0), (5,10) and (6,33).

2. Often data is expected to follow an exponential growth model of the form y = A^ekt, where t measures time and k is called the growth rate. By rewriting the equation as log y = k^t+logA, use this technique to find the values of A and k that give the best t of the exponential growth model to experimental data where the values of y at times 0, 1, 2 and 3 are 11, 23, 42 and 80 respectively.

Solution:

When data is collected from an experiment we would expect to have more accurate results by recording a large number of observations. This may result in a data set that is too large to use the method of Polynomial Interpolation. To find a polynomial of degree n, that fits a data set of more than n+1 points, we can use the method of Least Squares. Often, in this case, the graph of the polynomial will not lie on all the data points but it will be the best fit for the data.

Suppose we want to find a quadratic polynomial that fits the following data:

x	-2	3	5	6
y	25	0	10	33

We are finding a quadratic polynomial (degree n=2) for a data set of 4 points. Since $4 \neq n+1$ in this setting, we cannot use the method of Polynomial Interpolation. However, we can start the process in a similar manner. Each of the four points (x_1,y_1), (x_2,y_2), (x_3,y_3) and (x_4,y_4) can be substituted into the quadratic. Each of the data points must satisfy the standard quadratic equation:

$$y_1 = c_1 + c_2x + c_3x^2$$

For the same coefficients c_1, c_2 and c_3.

Then we can substitute each data point into this equation to get the following system:

$$c_1 + c_2(-2) + c_3(-2)^2 = 25$$

$$c_1 + 3c_2 + c_3(3)^2 = 0$$

$$c_1 + 5c_2 + c_3(5)^2 = 10$$

$$c_1 + 6c_2 + c_3(6)^2 = 33$$

Simplify the system by multiplying terms.

$$c_1 - 2c_2 + 4c_3 = 25$$

$$c_1 + 3c_2 + 9c_3 = 0$$

$$c_1 + 5c_2 + 25c_3 = 10$$

$$c_1 + 6c_2 + 36c_3 = 33$$

This system has more equations than unknowns and may not have a solution. Consider the matrix equation that corresponds to the system.

$$\begin{bmatrix} 1 & -2 & 4 \\ 1 & 3 & 9 \\ 1 & 5 & 25 \\ 1 & 6 & 36 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 25 \\ 0 \\ 10 \\ 33 \end{bmatrix}$$

Unlike the Vandermonde Matrix, the first matrix in the equation is not square and therefore has no inverse . At this point we will employ the transpose of this matrix to find the Normal Equations. This will enable us to find a solution.

Multiply each side of the matrix equation (left multiplication) by the transpose to get:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & 3 & 5 & 6 \\ 4 & 9 & 25 & 36 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 & 4 \\ 1 & 3 & 9 \\ 1 & 5 & 25 \\ 1 & 6 & 36 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & 3 & 5 & 6 \\ 4 & 9 & 25 & 36 \end{bmatrix} \cdot \begin{bmatrix} 25 \\ 0 \\ 10 \\ 33 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 12 & 74 \\ 12 & 74 & 360 \\ 74 & 360 & 2018 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 68 \\ 198 \\ 1538 \end{bmatrix}$$

Corresponding to this matrix equation are the Normal Equations:

$$4c_1 + 12c_2 + 74c_3 = 68$$

$$12c_1 + 74c_2 + 360c_3 = 198$$

$$74c_1 + 360c_2 + 2018c_3 = 1538$$

We solve this system and get the values for the coefficients c_1, c_2 and c_3.

Firstly we convert the equations into Ax=b matrix form.

$$\begin{bmatrix} 4 & 12 & 74 \\ 12 & 74 & 360 \\ 74 & 360 & 2018 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 68 \\ 198 \\ 1538 \end{bmatrix}$$

Then we calculate the determinant A.

$$\text{Determinant A} = \begin{bmatrix} 4 & 12 & 74 \\ 12 & 74 & 360 \\ 74 & 360 & 2018 \end{bmatrix} = 22472$$

After that we calculate the determinant for Ac_1.

$$\text{Determinant Ac}_1 = \begin{bmatrix} 68 & 12 & 74 \\ 198 & 74 & 360 \\ 1538 & 360 & 2018 \end{bmatrix} = 43800$$

Then we calculate the determinant for Ac_2.

$$\text{Determinant Ac}_2 = \begin{bmatrix} 4 & 68 & 74 \\ 12 & 198 & 360 \\ 74 & 1538 & 2018 \end{bmatrix} = -170136$$

Then we calculate the determinant Ac_3.

$$\text{Determinant Ac}_3 = \begin{bmatrix} 4 & 12 & 68 \\ 12 & 74 & 190 \\ 74 & 360 & 1538 \end{bmatrix} = 45872$$

Then we derive the solution set.

$$\text{c}_1 = \frac{\text{detAc}_1}{\text{detA}} = \frac{5475}{2809} = 1.9491$$

$$\text{c}_2 = \frac{\text{detAc}_2}{\text{detA}} = \frac{-21267}{2809} = -7.57102$$

$$\text{c}_3 = \frac{\text{detAc}_3}{\text{detA}} = \frac{5734}{2809} = 2.0413$$

Substitute into the origin equation find values. The resulting polynomial will be equal

$$\text{y}_1 = 1.9491 - 7.57102\text{x} + 2.0413\text{x}^2$$

We also can rewrite the obtained equation in general form.

$$y = 2.0413\text{x}^2 - 7.57102\text{x} + 1.9491$$

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Usually the problems, which is involving exponential decay and growth involve fitting a set of data. We have a function of the form $y(t) = ae^{kt}$, where a represents the initial quantity and k is called the growth or decay constant, in our case k it is the growth rate.

If we start with initial equation $y = ae^{kt}$ then we have

$$\ln(\text{y}) = \ln(\text{ae}^{\text{kt}})$$

Then we obtained the following.

$$\ln(\text{y}) = \ln(\text{a}) + \text{kt}\ln(\text{e})$$

We can simplify the equation.

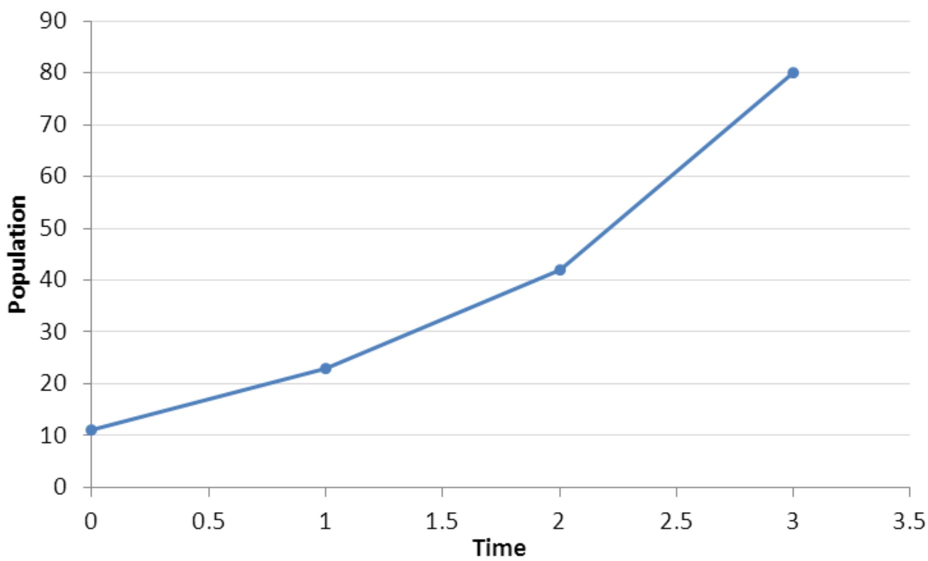
$$\ln(y) = \ln(a) + kt$$

We can see that if we plot t on the horizontal and ln(y) on the vertical then ln(y) is a linear function of t where k is the slope and ln(a) is the slope. We provide the givn set of population data in the Table.

Table

Time	Y(t)
0	11
1	23
2	42
3	80

Based on the information in the table we can represent the graph.



We see that the initial population (the population when t = 0) is 11. So, $P_0 = 11$. Thus, we need to solve the equation.

$$y(0) = ae^{kt}$$

We also know that

$$\ln(y) = \ln(a) + kt$$

If we put t = 0 we can see that value of $\ln(y)$ will be equal to ln(a). We also can write.

$$y(0) = ae^{k0}$$

$$y(0) = a$$

If we put t = 1, then we can find the value of a and k.

$$y(1) = ae^{k}$$

As we have find the value of a which is equal to 11, we now can find the value of k.

$$23 \ = \ 11e^{k}$$

$$\ln\left(\frac{23}{11}\right) = \ln(\mathrm{e})^{\mathrm{k}}$$

From there we can find the value of k.

$$k = \ln\left(\frac{23}{11}\right)$$

$$\mathrm{k} = 0.738$$

Then we consider the value of t = 2. Then k will be equal

$$k = \frac{\ln\left(\frac{42}{11}\right)}{2}$$

$$\mathrm{k} = 0.670$$

Then we consider the value of t = 3. Then k will be equal

$$k = \frac{\ln\left(\frac{80}{11}\right)}{3}$$

$$\mathrm{k} = 0.661$$

Thus the value of k = 0.690