Answer on Question #46954 – Math – Analytic Geometry

Problem.

Under what conditions on a, the sphere x^2+y^2+z^2+ax-y=0 and x^2+y^2+z^2+x+2z+1=0 intersect each other at an angle 45 degrees.

Solution.

The first sphere has equation

$$x^2 + y^2 + z^2 + ax - y = 0$$

or

$$\left(x+\frac{a}{2}\right)^2 + \left(y-\frac{1}{2}\right)^2 + z^2 = \frac{a^2}{4}+\frac{1}{4}.$$

Hence the first sphere has center $\left(-\frac{a}{2},\frac{1}{2},0\right)$ and radius $\sqrt{\frac{a^2}{4}+\frac{1}{4}}$.

The second sphere has equation

$$x^2 + y^2 + z^2 + x + 2z + 1 = 0$$

or

$$\left(x+\frac{1}{2}\right)^2 + y^2 + (z+1)^2 = \frac{1}{4}.$$

Hence the first sphere has center $\left(-\frac{1}{2},0,-1\right)$ and radius $\frac{1}{2}$.

Suppose that (x_0, y_0, z_0) is point from intersection of spheres. Therefore

$$x_0^2 + y_0^2 + z_0^2 + ax_0 - y_0 = 0$$

and

$$x_0^2 + y_0^2 + z_0^2 + x_0 + 2z_0 + 1 = 0.$$

The angle is between spheres is equal to the angle to tangent planes at point (x_0, y_0, z_0). The angle between planes is equal to angle be normal vector of this planes. The normal vectors of tangent planes at point (x_0, y_0, z_0) are $\left(x_0+\frac{a}{2}, y_0-\frac{1}{2}, z_0\right)$ and $\left(x_0+\frac{1}{2}, y_0, z_0+1\right)$ (this is the vectors from centers of the spheres to point) (x_0, y_0, z_0). Therefore the angle is between spheres is equal to the angle between vectors $\left(x_0+\frac{\alpha}{2}, y_0-\frac{1}{2}, z_0\right)$ and $\left(x_0+\frac{1}{2}, y_0, z_0+1\right)$. Hence

$$\frac{x_0^2+\frac{ax_0}{2}+\frac{x_0}{2}+\frac{a}{4}+y_0^2-\frac{y_0}{2}+z_0^2+z_0}{\sqrt{\left(x_0+\frac{a}{2}\right)^2+\left(y_0-\frac{1}{2}\right)^2+z_0^2}\cdot\sqrt{\left(x_0+\frac{1}{2}\right)^2+y_0^2+(z_0+1)^2}} = \cos 45^\circ = \frac{\sqrt{2}}{2}.$$

or

$$x_0^2+y_0^2+z_0^2+ax_0-y_0+x_0^2+y_0^2+z_0^2+x_0+2z_0 = \frac{\sqrt{2}}{2}\cdot\sqrt{\frac{a^2}{4}+\frac{1}{4}}-\frac{a}{4}.$$

Then

$$-1 = \frac{\sqrt{2}}{2}\cdot\sqrt{\frac{a^2}{4}+\frac{1}{4}}-\frac{a}{4}$$

or

$$a-4 = \sqrt{2(a^2+1)},$$

Therefore $\alpha > 4$ and

$$a^2-8a+16 = 2a^2+2,$$

$$a^2+8a-14 = 0,$$

$$\alpha = \frac{-8\pm\sqrt{64+4\cdot 8\cdot 14}}{2} = -4\pm\sqrt{16+8\cdot 14} = -4\pm 8\sqrt{2}.$$

Hence $a = 8\sqrt{2}-4$, as $a > 4$.

Answer. $a = 8\sqrt{2}-4$.