Answer on Question# #46817 – Mathematics – Calculus

Question:

Test the convergence of the series

$$\sum_{n=1}^{\infty}\left(\sqrt{n^2+1}-n\right)x^{2n}, x>0. \quad (1)$$

Solution:

Let's rewrite (1) in the following form

$$\sum_{n=1}^{\infty}\left(\sqrt{n^2+1}-n\right)x^{2n}=\sum_{n=1}^{\infty}u_n(x), \quad (2)$$

To determine the interval of convergence for the series (2) we take absolute values and apply the Ratio Test:

$$\lim_{n\to\infty}\left|\frac{u_{n+1}(x)}{u_n(x)}\right|=\lim_{n\to\infty}\left|\frac{\left(\sqrt{(n+1)^2+1}-(n+1)\right)x^{2(n+1)}}{\left(\sqrt{n^2+1}-n\right)x^{2n}}\right|=\lim_{n\to\infty}\frac{\left(\sqrt{(n+1)^2+1}-(n+1)\right)}{\left(\sqrt{n^2+1}-n\right)}\left|\frac{x^{2n}x^2}{x^{2n}}\right|$$

$$=|x^2|\lim_{n\to\infty}\frac{\frac{1}{2}\frac{2(n+2)}{\sqrt{(n+1)^2+1}}-1}{\frac{n}{\sqrt{n^2+1}}-1}=|x^2|\lim_{n\to\infty}\frac{\frac{1}{((n+1)^2+1)^{3/2}}}{\frac{1}{(n^2+1)^{3/2}}}$$

$$=|x^2|\lim_{n\to\infty}\frac{(n^2+1)^{3/2}}{((n+1)^2+1)^{3/2}}=|x^2|\left(\lim_{n\to\infty}\frac{n^2+1}{(n+1)^2+1}\right)^{3/2}$$

$$=|x^2|\left(\lim_{n\to\infty}\frac{n^2\left(1+\frac{1}{n^2}\right)}{n^2\left(\left(1+\frac{1}{n}\right)^2+\frac{1}{n^2}\right)}\right)^{3/2}=|x^2|\left(\lim_{n\to\infty}\frac{\left(1+\frac{1}{n^2}\right)}{\left(\left(1+\frac{1}{n}\right)^2+\frac{1}{n^2}\right)}\right)^{3/2}=|x^2|1^{3/2}$$

$$=|x^2|.$$

Note that for calculating the limit we have used the properties of limits. Hence, the series converges for $|x|<1$ (or -1<x<1) and diverges for $|x|>1$ (or x<-1 and x>1).

Now let`s test the convergence of the series at the endpoints of the interval separately.

At x=-1 the series is

$$\sum_{n=1}^{\infty}\left(\sqrt{n^2+1}-n\right)(-1)^{2n}=\sum_{n=1}^{\infty}\left(\sqrt{n^2+1}-n\right)=\left(\sqrt{2}-1\right)+\left(\sqrt{5}-2\right)+\cdots. \quad (3)$$

It easy to see, that the series (1) diverges at x=-1. It is clearly that for all natural number *n* we always receive: $(-1)^{2n}=1$.

At x=1 the series is

$$\sum_{n=1}^{\infty}\left(\sqrt{n^2+1}-n\right)(1)^{2n}=\sum_{n=1}^{\infty}\left(\sqrt{n^2+1}-n\right)=\left(\sqrt{2}-1\right)+\left(\sqrt{5}-2\right)+\cdots. \quad (4)$$

The series (1) also diverges at x=1. Perhaps, is not quite obvious that the series diverges. So, let`s use the limit comparison test.

First we rewrite the original series (4) in the following convenient form:

$$\sum_{n=1}^{\infty}\left(\sqrt{n^2+1}-n\right)=\sum_{n=1}^{\infty}\frac{(\sqrt{n^2+1}-n)(\sqrt{n^2+1}+n)}{\sqrt{n^2+1}+n}=\sum_{n=1}^{\infty}\frac{(n^2+1-n^2)}{\sqrt{n^2+1}+n}=\sum_{n=1}^{\infty}\frac{1}{\sqrt{n^2+1}+n}.$$

Then we compare it with the divergent harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$:

$$\lim_{n\to\infty}\frac{\frac{1}{n}}{\frac{1}{\sqrt{n^2+1+n}}}=\lim_{n\to\infty}\frac{\sqrt{n^2+1}+n}{n}=\lim_{n\to\infty}\sqrt{1+\frac{1}{n^2}}+1=2>0$$

As we see, according to the limit comparison test, the original series (4) is also divergent.

Therefore, the series diverges for $x \le -1$ and for $x \ge 1$. It converges for -1 < x < 1

Answer: The series $\sum_{n=1}^{\infty}(\sqrt{n^2+1}-n)x^{2n},\ \ x>0$ converges for -1 < x < 1 and diverges for $x \le -1$ and for $x \ge 1$.