Answer on Question #46798 – Math – Analytic Geometry

Question:

Find two unit vectors perpendicular to both $\overline{\boldsymbol{A}} = \overline{\boldsymbol{i}} - 2\overline{\boldsymbol{j}} + 3\overline{\boldsymbol{k}}$ and $\overline{\boldsymbol{B}} = -2\overline{\boldsymbol{i}} + 4\overline{\boldsymbol{j}}$.

Solution.

There are two ways to construct vectors perpendicular to the pair of given vectors: through scalar product and through cross (or vector) product.

1. Scalar (dot, inner) product.

Let $\overline{\boldsymbol{C}} = x\overline{\boldsymbol{i}} - y\overline{\boldsymbol{j}} + z\overline{\boldsymbol{k}}$ be the vector perpendicular to both given vectors.

Then, by properties of scalar product $\overline{\boldsymbol{A}} \cdot \overline{\boldsymbol{C}} = 0$ and $\overline{\boldsymbol{B}} \cdot \overline{\boldsymbol{C}} = 0$.

It means that each solution of system $\begin{cases} x - 2y + 3z = 0 \\ -2x + 4y = 0 \end{cases}$ gives the required vector. To

solve this system, set, for example, y= 1 and find other coordinates from system

$\begin{cases} x - 2 + 3z = 0 \\ -2x + 4 = 0 \end{cases}$

From second equation we have $x = 2$, and then obtain $z = 0$ from the first equation

Hence $\overline{\boldsymbol{C}} = 2\overline{\boldsymbol{i}} + \overline{\boldsymbol{j}}$ is perpendicular to both $\overline{\boldsymbol{A}}$ and $\overline{\boldsymbol{B}}$.

To obtain unit vectors we have to divide the vector by its magnitude:

$$\overline{\boldsymbol{n}}_{1,2} = \pm\frac{\overline{\boldsymbol{C}}}{|\overline{\boldsymbol{C}}|} = \pm\frac{2\overline{\boldsymbol{i}} + \overline{\boldsymbol{j}}}{\sqrt{4+1}} = \pm\left(\frac{2}{\sqrt{5}}\overline{\boldsymbol{i}} + \frac{1}{\sqrt{5}}\overline{\boldsymbol{j}}\right)$$

2. Cross product

By properties of cross product $\overline{\boldsymbol{C}} = \overline{\boldsymbol{A}} \times \overline{\boldsymbol{B}}$ is perpendicular to both $\overline{\boldsymbol{A}}$ and $\overline{\boldsymbol{B}}$.
Calculate

$$\overline{\boldsymbol{C}} = \begin{vmatrix} \overline{\boldsymbol{i}} & \overline{\boldsymbol{j}} & \overline{\boldsymbol{k}} \\ 1 & -2 & 3 \\ -2 & 4 & 0 \end{vmatrix} = \begin{vmatrix} -2 & 3 \\ 4 & 0 \end{vmatrix}\overline{\boldsymbol{i}} - \begin{vmatrix} 1 & 3 \\ -2 & 0 \end{vmatrix}\overline{\boldsymbol{j}} + \begin{vmatrix} 1 & -2 \\ -2 & 4 \end{vmatrix}\overline{\boldsymbol{k}} = -12\overline{\boldsymbol{i}} - 6\overline{\boldsymbol{j}}$$

After division by magnitude we obtain the same result

$$\overline{\boldsymbol{n}}_{1,2} = \pm\frac{\overline{\boldsymbol{C}}}{|\overline{\boldsymbol{C}}|} = \pm\frac{-12\overline{\boldsymbol{i}} - 6\overline{\boldsymbol{j}}}{\sqrt{144+36}} = \pm\frac{-12\overline{\boldsymbol{i}} - 6\overline{\boldsymbol{j}}}{6\sqrt{5}} = \mp\left(\frac{2}{\sqrt{5}}\overline{\boldsymbol{i}} + \frac{1}{\sqrt{5}}\overline{\boldsymbol{j}}\right)$$

Answer: Two unit vectors are $\overline{\boldsymbol{n}}_1 = \frac{2}{\sqrt{5}}\overline{\boldsymbol{i}} + \frac{1}{\sqrt{5}}\overline{\boldsymbol{j}}$ and $\overline{\boldsymbol{n}}_2 = -\frac{2}{\sqrt{5}}\overline{\boldsymbol{i}} - \frac{1}{\sqrt{5}}\overline{\boldsymbol{j}}$