Answer on Question #46676 – Math – Algorithms | Quantitative Methods

Solve the system of equations using LU decomposition method.

$$5x - 2y + z = 4$$

$$7x + y - 5z = 6$$

$$3x + 7y + 4z = 10$$

Solution:

Suppose we have the system of equations

$$AX = B$$

The motivation for an LU decomposition is based on the observation that systems of equations involving triangular coefficient matrices are easier to deal with. Indeed, the whole point of Gaussian Elimination is to replace the coefficient matrix with one that is triangular. The LU decomposition is another approach designed to exploit triangular systems.

We suppose that we can write

$$A = LU$$

Where L is a lower triangular matrix and U is an upper triangular matrix. Our aim is to find L and U and once we have done so we have found an LU decomposition of A. So we can note the rule for this method of solution the system of equations.

An LU decomposition of a matrix A is the product of a lower triangular matrix and an upper triangular matrix that is equal to A.

Based on the above information we solve system of equations by using the decomposition method.

Convert the equations into $AX = B$ matrix form.

$$\begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 10 \end{bmatrix}$$

Then we find the determinant of a matrix A.

$$\det A = \begin{vmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{vmatrix} = 20 + 30 + 49 - (-175) - (-56) - 3 = 327$$

We keep transforming A until lower-zero-leading form is derived.

$$\begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix} => \begin{bmatrix} 5 & -2 & 1 \\ (7-(7)) & (1-\left(-\frac{14}{5}\right)) & (-5-\left(\frac{7}{5}\right)) \\ (3-(3)) & (7-\left(-\frac{6}{5}\right)) & (4-\frac{3}{5}) \end{bmatrix} => \begin{bmatrix} 5 & -2 & 1 \\ 0 & \frac{19}{5} & -\frac{32}{5} \\ 0 & \frac{41}{5} & \frac{17}{5} \end{bmatrix}$$

Now we consider the following matrix.

$$\begin{bmatrix} 5 & -2 & 1 \\ 0 & \frac{19}{5} & -\frac{32}{5} \\ 0 & \frac{41}{5} & \frac{17}{5} \end{bmatrix} => \begin{bmatrix} 5 & -2 & 1 \\ 0 & \frac{19}{5} & -\frac{32}{5} \\ 0 & (\frac{41}{5}-\frac{41}{5}) & (\frac{17}{5}-(-\frac{1312}{95})) \end{bmatrix} => \begin{bmatrix} 5 & -2 & 1 \\ 0 & \frac{19}{5} & -\frac{32}{5} \\ 0 & 0 & \frac{327}{19} \end{bmatrix}$$

The resulting matrix is the upper triangual matrix U. Lower triangular matrix L will built from an identity matrix complemented with multipliers.

$$\mathrm{A} = \mathrm{LU} = \begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{7}{5} & 1 & 0 \\ \frac{3}{5} & \frac{41}{19} & 1 \end{bmatrix} \begin{bmatrix} 5 & -2 & 1 \\ 0 & \frac{19}{5} & -\frac{32}{5} \\ 0 & 0 & \frac{327}{19} \end{bmatrix}$$

Now we solve $\mathrm{LY} = \mathrm{B}$ for Y.

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{7}{5} & 1 & 0 \\ \frac{3}{5} & \frac{41}{19} & 1 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 10 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{3}{5} & \frac{41}{19} & 1 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix} => \begin{bmatrix} 4 \\ \frac{2}{5} \\ 10 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{41}{19} & 1 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix} => \begin{bmatrix} 4 \\ \frac{2}{5} \\ \frac{38}{5} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix} => \begin{bmatrix} 4 \\ \frac{2}{5} \\ \frac{128}{19} \end{bmatrix}$$

Then we solve $\mathrm{UX} = \mathrm{Y}$ for X.

$$\begin{bmatrix} 5 & -2 & 1 \\ 0 & \frac{19}{5} & -\frac{32}{5} \\ 0 & 0 & \frac{327}{19} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ \frac{2}{5} \\ \frac{128}{19} \end{bmatrix}$$

$$\begin{bmatrix} 5 & -2 & 1 \\ 0 & \frac{19}{5} & -\frac{32}{5} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} => \begin{bmatrix} 4 \\ \frac{2}{5} \\ \frac{128}{327} \end{bmatrix}$$

$$\begin{bmatrix} 5 & -2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} => \begin{bmatrix} 4 \\ \frac{250}{327} \\ \frac{128}{327} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} => \begin{bmatrix} \frac{112}{109} \\ \frac{250}{327} \\ \frac{128}{327} \end{bmatrix}$$

Finally we represent the solution set

$$x = \frac{112}{109} \approx 1.027523$$
$$y = \frac{250}{327} \approx 0.764526$$
$$z = \frac{128}{327} \approx 0.391437$$