Answer on Question #46554 – Math – Differential Calculus

Find all the roots of the equation $x^3 + 6x^2 + 11x + 6 = 0$ by the Graeffe's root squaring method using three squaring.

Solution:

Graeffe's method is one of the roots finding method of a polynomial with real coefficients. This method gives all the roots approximated in each iteration also this is one of the direct roots finding method. This method does not require any initial guesses for roots.

For our polynomial we can estimate the value of the roots by evaluating 2^i root of

$$\left|\frac{a_i}{a_{i-1}}\right|, \qquad i = 1,2, \dots n$$

Where n is the degree of the given polynomial. We have a polynomial of degree 3 so I will be equal to 3.

Consider the polynomial: $P(x) = x^3 + 6x^2 + 11x + 6 = 0$ which is a polynomial of degree 3 and has the roots -1, -2, -3. Then the possible Rational Roots the following solutions: $\pm1,\ \pm2,\ \pm3,\ \pm6$.

It can be easily verified that $P(x) = x^3 + 6x^2 + 11x + 6 = (x+1)(x+2)(x+3)$.

Then we collect all even terms on one side and all odd terms on the other and then squaring yields.

$$(x^3 + 11x)^2 = (6x^2 + 6)^2$$

We obtained the following.

$$x^6 + 22x^4 + 121x^2 = 36x^4 + 72x^2 + 36$$

Simplify the equation.

$$x^6 + 22x^4 + 121x^2 - 36x^4 - 72x^2 - 36 = x^6 - 14x^4 + 49x^2 - 36$$

We write $y = x^2$ in equation above, gives us a polynomial p(y) as given below which is of same degree as p(x).

$$p(y) = y^3 - 14y^2 + 49y - 36$$

The relationship between the roots of p(x) and p(y) can be shown that in general if

$$p(x) = (x-a)(x-b)(x-c)$$

Then we have

$$p(y) = (x-a^2)(x-b^2)(x-c^2)$$

Now we shall refer to collect all even terms on one side and all odd terms on the other and then write $y = x^2$.

$$(x^3 + 49x)^2 = (14x^2 + 36)^2$$

We obtained the following.

$$x^6 + 98x^4 + 2401x^2 = 196x^4 + 1008x^2 + 1296$$

Simplify the equation.

$$x^6 + 98x^4 + 2401x^2 - 196x^4 - 1008x^2 - 1296 = x^6 - 98x^4 + 1393x^2 - 1296$$

We repeat root squaring successively and obtained a series of polynomials, all of degree 3, as shown below:

$$y^3 - 98y^2 + 1393y - 1296$$

Finally we find for i = 3, the polynomial will be.

$$(x^3 + 1393x)^2 = (98x^2 + 1296)^2$$

We obtained the following.

$$x^6 + 2786x^4 + 1\,940\,499x^2 = 9604x^4 + 254016x^2 + 1\,679\,616$$

Simplify the equation.

$$x^6 + 2786x^4 + 1\,940\,499x^2 - 9604x^4 - 254016x^2 - 1\,679\,616$$
$$= x^6 - 6818x^4 + 1\,686\,483x^2 - 1\,679\,616$$

We repeat root squaring successively and obtained a series of polynomials, all of degree 3, as shown below:

$$y^3 - 6818y^2 + 1\,686\,433y - 1\,679\,616$$

Once the roots get sufficiently separated, it is easy to extract all the 3 roots from the last polynomial. The algorithm assumes for the time being that the initial roots of p(x) were all distinct, say |a| > |b| > |c|.

Now we can calculate the roots. The roots of polynomial are

$$\sqrt{\frac{36}{49}} = 0.857143$$

$$\sqrt{\frac{49}{14}} = 1.870829$$

$$\sqrt{\frac{14}{1}} = 3.741657$$

Similarly we calculate the roots of polynomial (2) are

$$\sqrt[4]{\frac{1296}{1393}} = 0.98211$$

$$\sqrt[4]{\frac{1393}{98}} = 1.941696$$

$$\sqrt[4]{\frac{98}{1}} = 3.146346$$

Finally we note the estimation of the roots obtained from polynomial (3).

$$\sqrt[8]{\frac{1\ 679\ 616}{1\ 686\ 433}} = 0.999493822$$

$$\sqrt[8]{\frac{1686\ 433}{6818}} = 1.991425261$$

$$\sqrt[8]{\frac{6818}{1}} = 3.014443336$$

Finally by inserting possible Rational Roots in the original equation we get as before

$$x_1 = -1\,, x_2 = -2\,, x_3 = -3$$