Answer on Question #46538 – Math – Statistics and Probability

The joint density function for continuous random variable (X, Y) is given by

$$f_{XY}(x,y) = \begin{cases} 6e^{-(2x+3y)}, & x \geq 0, y \geq 0 \\ 0, & e.w \end{cases}$$

Find

(a). $P[0 \leq X + Y \leq 1]$

(b). Find marginal densities for X and Y respectively. Are X and Y independent random variables? Justify your answer.

Solution

(a)

$$P[0 \leq X + Y \leq 1] = \int_0^1 dx \int_0^{1-x} dy\, 6e^{-(2x+3y)} = \int_0^1 dx\, 2e^{-2x} \int_0^{1-x} dy\, 3e^{-3y}.$$

$$\int_0^{1-x} dy\, 3e^{-3y} = \int_0^{1-x} d(3y)\, e^{-3y} = \int_0^{3(1-x)} dz\, e^{-z},$$

where $z = 3y$.

$$\int_0^{3(1-x)} dz\, e^{-z} = -e^{-z}\Big|_0^{3(1-x)} = e^0 - e^{-3(1-x)} = 1 - e^{3x-3}.$$

$$P[0 \leq X + Y \leq 1] = \int_0^1 dx\, 2e^{-2x}(1 - e^{3x-3}) = \int_0^1 dx\, 2e^{-2x} - \int_0^1 dx\, 2e^{x-3}.$$

$$\int_0^1 dx\, 2e^{-2x} = e^0 - e^{-2} = 1 - \frac{1}{e^2}.$$

$$\int_0^1 dx\, 2e^{x-3} = \frac{2}{e^3}\int_0^1 dx\, e^x = \frac{2}{e^3}(e-1) = \frac{2}{e^2} - \frac{2}{e^3}.$$

$$P[0 \leq X + Y \leq 1] = 1 - \frac{1}{e^2} - \left(\frac{2}{e^2} - \frac{2}{e^3}\right) = 1 - \frac{3}{e^2} + \frac{2}{e^3} = 0.69.$$

(b)

$$f_X(x) = \int_0^\infty dy\, 6e^{-(2x+3y)} = 2e^{-2x}\int_0^\infty dy\, 3e^{-(3y)} = 2e^{-2x}\left(-e^{-3y}\Big|_0^\infty\right) = 2e^{-2x}(1-0) = 2e^{-2x}.$$

$$f_Y(y) = \int_0^\infty dx\, 6e^{-(2x+3y)} = 3e^{-3y}\int_0^\infty dx\, 2e^{-(2x)} = 3e^{-3y}\left(-e^{-2x}\Big|_0^\infty\right) = 3e^{-3y}(1-0) = 3e^{-3y}.$$

$$f_X(x) \cdot f_Y(y) = 2e^{-2x} \cdot 3e^{-3y} = 6e^{-(2x+3y)} = f_{XY}(x,y).$$

Therefore X and Y are independent random variables.