Answer on Question #46245 – Math – Statistics and Probability

Verify that $f(x,y) = xye^{-x}e^{-y}, x > 0, y > 0$, satisfies the conditions necessary to be density for a continuous random variables x and y. Are X and Y independent? Find the correlation coefficient.

Solution

In order for a function $f(x,y)$ to be a joint density it must satisfy $\int_0^\infty dx \int_0^\infty dy\, f(x,y) \ = 1$ and

$f(x,y) \geq 0$.

It is easy to see that $xye^{-x}e^{-y} \geq 0$ when $x > 0, y > 0$.

We need to find

$$I = \int_0^\infty dx \int_0^\infty dy\, xye^{-x}e^{-y} = \int_0^\infty xe^{-x}dx \cdot \int_0^\infty ye^{-y}dy = I_1 \cdot I_2.$$

We can see that $I_1 = I_2$ because we can obtain it one from another by transformations $x \leftrightarrow y$.

So $I = I_1^2$. Then

$$I_1 = \int_0^\infty xe^{-x}dx = \int_0^\infty xd(-e^{-x}) = -e^{-x}x|_0^\infty - \int_0^\infty (-e^{-x})dx = 0 + \int_0^\infty e^{-x}dx = -e^{-x}|_0^\infty = 1.$$

Therefore $I = I_1^2 = 1$ and $f(x,y)$ satisfies the conditions necessary to be density for a continuous random variables x and y.

The marginal densities are given by

$$f_X(x) = \int_0^\infty dy\, xye^{-x}e^{-y} = xe^{-x}\int_0^\infty ye^{-y}dy = xe^{-x} \cdot 1 = xe^{-x},$$

$$f_Y(y) = \int_0^\infty dx\, xye^{-x}e^{-y} = ye^{-y}\int_0^\infty xe^{-x}dx = ye^{-y} \cdot 1 = ye^{-y}.$$

$$f_X(x)f_Y(y) = xe^{-x}ye^{-y} = xye^{-x}e^{-y} = f(x,y).$$

That's why X and Y are independent.

The correlation coefficient $\rho(X,Y)$ defined by

$$\rho(X,Y) = \frac{Cov(x,y)}{\sqrt{Var(X)Var(Y)}},$$

where $Var(X)$ is variance of a random variable X, $Var(Y)$ is variance of a random variable Y, $Cov(x,y)$ is he covariance of two random variables X and Y. But if X and Y are independent, then $Cov(x,y) = 0$. So

$$\rho(X,Y) = 0.$$