Answer on Question #46204 – Math – Analytic Geometry

Problem.

Find the path traced by the centre of the sphere which touches the lines
x/y = y/(-1) = z/2 and 2x = y , y-z = 0.

Solution:

We will rewrite lines $\frac{x}{y} = \frac{y}{-1} = \frac{z}{2}$ and $2x = y, y - z = 0$ in the parametric form

Let $t \in \mathbb{R}$ is the parameter of the first line and $\frac{x}{y} = \frac{y}{-1} = \frac{z}{2} = t$. Then $x = -t^2, y = -t, z = 2t$.

Hence $\frac{x}{y} = \frac{y}{-1} = \frac{z}{2} = t$ isn't a line it is curve.

Let $s \in \mathbb{R}$ is the parameter of the second line and $x = s$. Then $x = s, y = 2s, z = 2s$.

The tangent vector of the first $(-2t, -1,2)$.

The direction vectors of the second line is $(1,2,2)$.

Suppose that (x_0, y_0, z_0) is center of the sphere. Then

$$\sqrt{(x_0 - t^2)^2 + (y_0 + t)^2 + (z_0 - 2t)^2} = \sqrt{(x_0 - s)^2 + (y_0 - 2s)^2 + (z_0 - 2s)^2},$$

as the radiuses of the sphere.

If

$$\sqrt{(x_0 - t^2)^2 + (y_0 + t)^2 + (z_0 - 2t)^2} = \sqrt{(x_0 - s)^2 + (y_0 - 2s)^2 + (z_0 - 2s)^2},$$

then

$$x_0^2 - 2x_0 t^2 + t^4 + y_0^2 + 2y_0 t + t^2 + z_0 - 4z_0 t + 4t^2$$
$$= x_0 - 2x_0 s + s^2 + y_0^2 - 4y_0 s + 4s^2 + z_0^2 - 4z_0 s + 4s^2$$

or

$$-2t(x_0 t - y_0 + 2z_0) + t^4 + 5t^2 = -2s(x_0 + 2y_0 + 2z_0) + 9s^2.$$

The sphere should touches the lines (the direction vector of the curves should be perpendicular to the vector from the center of the sphere to the point on the line), so

$$(x_0 + t^2) \cdot (-2t) + (y_0 + t) \cdot (-1) + (z_0 - 2t) \cdot 2 = 0$$

and

$$(x_0 - s) \cdot 1 + (y_0 - 2s) \cdot 2 + (z_0 - 2s) \cdot 2 = 0.$$

Therefore

$$-2tx_0 - y_0 + 2z_0 = 2t^3 + 5t \text{ or } -y_0 + 2z_0 = 2t^3 + 5t + 2tx_0$$

and

$$x_0 + 2y_0 + 2z_0 = 9s.$$

Therefore

$$-2t(x_0 t + 2t^3 + 5t + 2tx_0) + t^4 + 5t^2 = -18s^2 + 9s^2$$

or

$$-6tx_0 - 3t^4 - 5t^2 = -9s^2.$$

Hence

$x_0 = \frac{9s^2 - 3t^4 - 5t^2}{6t}$, $\quad 2y_0 + 2z_0 = 9s - \frac{9s^2 - 3t^4 - 5t^2}{6t}$,

$$-y_0 + 2z_0 = 2t^3 + 5t + \frac{9s^2 - 3t^4 - 5t^2}{3}$$

Then $y_0 = 9s + 2t^3 + 5t - \frac{9s^2 - 3t^4 - 5t^2}{6t} + \frac{9s^2 - 3t^4 - 5t^2}{3}$ and $z_0 = 2t^3 + 5t + \frac{9}{2}s - \frac{9s^2 - 3t^4 - 5t^2}{12t} + \frac{9s^2 - 3t^4 - 5t^2}{3}$.

Answer:

$x_0 = \frac{9s^2 - 3t^4 - 5t^2}{6t}$, $y_0 = 9s + 2t^3 + 5t - \frac{9s^2 - 3t^4 - 5t^2}{6t} + \frac{9s^2 - 3t^4 - 5t^2}{3}$, $z_0 = 2t^3 + 5t + \frac{9}{2}s - \frac{9s^2 - 3t^4 - 5t^2}{12t} + \frac{9s^2 - 3t^4 - 5t^2}{3}$.