Answer on Question #46181 – Math - Vector Calculus

Problem.

Find the angle between the lines whose direction cosines are given by the equations.
(i) l + 2m +3n = 0 and 3lm – 4ln + mn = 0
(ii) l + m + n = 0 and l2 + m2 – n2 = 0
Remark: I suppose that the statement is incorrectly formatted. The correct statement is:
"Find the angle between the lines whose direction cosines are given by the equations.
(i) $l+2m+3n=\ 0$ and $3lm-4ln+mn=0$
(ii) $l+m+n=0$ and $l^2+m^2-\ n^2=0$"

Solution:

(i) If l,m,n are direction cosines, then $l^2+m^2+n^2=1$.
$l=-2m-3n$, so $3(-2m-3n)m-4(-2m-3n)n+mn=0$. Hence $-6m^2-9mn+8mn+12n^2+mn=0$ and $12n^2=6m^2$ or $2n^2=m^2$. Then $m=\sqrt{2}n$ and $m=-\sqrt{2}n$.
$l^2+m^2+n^2=1$ and $l=-2m-3n$, so $4m^2+12mn+9n^2+m^2+n^2=1$. Hence $5m^2+12mn+10n^2=20n^2+12mn=1$.
If $m=\sqrt{2}n$, then $20n^2+12\sqrt{2}n^2=1$ or $n^2(20+12\sqrt{2})=1$. Then $n=\frac{1}{\sqrt{20+12\sqrt{2}}}$ or $n=-\frac{1}{\sqrt{20+12\sqrt{2}}}$. We obtain two triples (l,m,n) of direction cosines $\left(\frac{-2\sqrt{2}-3}{\sqrt{20+12\sqrt{2}}},\frac{\sqrt{2}}{\sqrt{20+12\sqrt{2}}},\frac{1}{\sqrt{20+12\sqrt{2}}}\right)$, $\left(\frac{2\sqrt{2}+3}{\sqrt{20+12\sqrt{2}}},-\frac{\sqrt{2}}{\sqrt{20+12\sqrt{2}}},-\frac{1}{\sqrt{20+12\sqrt{2}}}\right)$.
If $m=-\sqrt{2}n$, then $20n^2-12\sqrt{2}n^2=1$ or $n^2(20-12\sqrt{2})=1$. Then $n=\frac{1}{\sqrt{20-12\sqrt{2}}}$ or $n=-\frac{1}{\sqrt{20-12\sqrt{2}}}$. We obtain two triples (l,m,n) of direction cosines $\left(\frac{2\sqrt{2}-3}{\sqrt{20-12\sqrt{2}}},-\frac{\sqrt{2}}{\sqrt{20-12\sqrt{2}}},\frac{1}{\sqrt{20-12\sqrt{2}}}\right)$, $\left(\frac{-2\sqrt{2}+3}{\sqrt{20-12\sqrt{2}}},\frac{\sqrt{2}}{\sqrt{20-12\sqrt{2}}},-\frac{1}{\sqrt{20-12\sqrt{2}}}\right)$.
Answer: $\left(\frac{-2\sqrt{2}-3}{\sqrt{20+12\sqrt{2}}},\frac{\sqrt{2}}{\sqrt{20+12\sqrt{2}}},\frac{1}{\sqrt{20+12\sqrt{2}}}\right)$, $\left(\frac{2\sqrt{2}+3}{\sqrt{20+12\sqrt{2}}},-\frac{\sqrt{2}}{\sqrt{20+12\sqrt{2}}},-\frac{1}{\sqrt{20+12\sqrt{2}}}\right)$, $\left(\frac{2\sqrt{2}-3}{\sqrt{20-12\sqrt{2}}},-\frac{\sqrt{2}}{\sqrt{20-12\sqrt{2}}},\frac{1}{\sqrt{20-12\sqrt{2}}}\right)$, $\left(\frac{-2\sqrt{2}+3}{\sqrt{20-12\sqrt{2}}},\frac{\sqrt{2}}{\sqrt{20-12\sqrt{2}}},-\frac{1}{\sqrt{20-12\sqrt{2}}}\right)$.

(ii) If l,m,n are direction cosines, then $l^2+m^2+n^2=1$. If $l^2+m^2=\ n^2$ and $l^2+m^2+n^2=1$, then $2n^2=1$ and $l^2+m^2=\frac{1}{2}$. $l+m=-n$, so $l^2+2lm+m^2=n^2$. Hence $2lm=0$ ($l=0$ or $m=0$) and $n^2=\frac{1}{2}$.
If $l=0$, then $m=\frac{1}{\sqrt{2}}$ and $n=-\frac{1}{\sqrt{2}}$ or $m=-\frac{1}{\sqrt{2}}$ and $n=\frac{1}{\sqrt{2}}$.
If $m=0$, then $n=\frac{1}{\sqrt{2}}$ and $l=-\frac{1}{\sqrt{2}}$ or $l=-\frac{1}{\sqrt{2}}$ and $n=\frac{1}{\sqrt{2}}$.
Threfore, we obtain three four triples (l,m,n) of direction cosines $\left(0,\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$, $\left(0,-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$, $\left(-\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\right)$, $\left(\frac{1}{\sqrt{2}},0,-\frac{1}{\sqrt{2}}\right)$.
Answer: $\left(0,\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$, $\left(0,-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$, $\left(-\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\right)$, $\left(\frac{1}{\sqrt{2}},0,-\frac{1}{\sqrt{2}}\right)$.