Answer on Question #46099-Math-Analytic Geometry

(a) Show that the angle between the two lines in which the plane $x - y + 2z = 0$ intersects the cone $x^2 + y^2 - 4z^2 + 6yz = 0$ is $\tan^{-1}\frac{\sqrt{6}}{7}$.

(b) Using projection show that the line passing through (-1, 8, 8) and (6, 2, 0) is perpendicular to the line passing through (4, 2, 3) and (2, 1, 2).

(c) At what point the origin must be shifted so that linear terms in the conicoid $x^2 + 2y^2 - z^2 - 2yz + 2xz + x - 3y + z + 4 = 0$ vanish? Justify.

Solution

(a) $x - y + 2z = 0 \to x = y - 2z.$

$$(y - 2z)^2 + y^2 - 4z^2 + 6yz = 0 \to y^2 + 4z^2 - 4yz + y^2 - 4z^2 + 6yz = 0 \to 2y^2 + 2yz = 0$$
$$\to y(y + z) = 0.$$

First case $y = -z$:

$$x = -z - 2z = -3z.$$

The equation of a line:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = t\begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix}.$$

Second case $y = 0$:

$$x = -2z.$$

The equation of a line:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = t\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}.$$

The angle between the two lines

$$\cos\theta = \frac{-3(-2) - 1\cdot 0 + 1\cdot 1}{\sqrt{(-3)^2 + (-1)^2 + (1)^2}\sqrt{(-2)^2 + (0)^2 + (1)^2}} = \frac{7}{\sqrt{55}};\ \sin\theta = \sqrt{1 - \left(\frac{7}{\sqrt{55}}\right)^2} = \sqrt{\frac{6}{55}}.$$

$$\theta = \tan^{-1}\frac{\sqrt{\frac{6}{55}}}{\frac{7}{\sqrt{55}}} = \tan^{-1}\frac{\sqrt{6}}{7}.$$

(b) The displacement vector of the first line is

$$\bar{v} = \begin{pmatrix} 6 - (-1) \\ 2 - 8 \\ 0 - 8 \end{pmatrix} = \begin{pmatrix} 7 \\ -6 \\ -8 \end{pmatrix}.$$

The displacement vector of the second line is

$$\bar{s} = \begin{pmatrix} 4 - 2 \\ 2 - 1 \\ 3 - 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}.$$

The orthogonal projection of $\bar{v}$ onto the second line is

$$\frac{\begin{pmatrix}7\\-6\\-8\end{pmatrix}\cdot\begin{pmatrix}2\\1\\1\end{pmatrix}}{\begin{pmatrix}2\\1\\1\end{pmatrix}\cdot\begin{pmatrix}2\\1\\1\end{pmatrix}}\begin{pmatrix}2\\1\\1\end{pmatrix}=\frac{7\cdot 2-6\cdot 1-8\cdot 1}{2^2+1^2+1^2}\begin{pmatrix}2\\1\\1\end{pmatrix}=\frac{0}{6}\begin{pmatrix}2\\1\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}.$$

That's why these lines are perpendicular to each other.

(c) Suppose that the origin is shifted to point (a, b, c), (x, y, z) are coordinates in old system, (x', y', z') are coordinates in new system. Then $x = x' + a, y = y' + b, z = z' + c$. After substitution this into $x^2 + 2y^2 - z^2 - 2yz + 2xz + x - 3y + z + 4 = 0$ we will obtain

$$(x')^2 + 2x'a + a^2 + 2(y')^2 + 4y'b + 2b^2 - (z')^2 - 2z'c - c^2 - 2y'z' - 2y'c - 2z'b - 2bc + 2x'z' + 2x'c + 2z'a + 2ac + x' + a - 3y' - 3b + z' + c + 4 = 0$$

or

$$(x')^2 + 2(y')^2 - (z')^2 - 2y'z' + 2x'z' + x'(2a + 2c + 1) + y'(4b - 2c - 3) - z'(2b + 2c - 2a - 1) + a^2 + 2b^2 - c^2 - 2bc + 2ac + a - 3b + c + 4 = 0.$$

The linear terms should vanish, so

$$\begin{cases} 2a + 2c + 1 = 0 \\ 4b - 2c - 3 = 0 \\ 2c + 2b - 2a - 1 = 0 \end{cases} \rightarrow \begin{cases} 2a + 2c + 1 = 0 \\ 4b - 2c - 3 = 0 \\ (2c + 2b - 2a - 1) + (2a + 2c + 1) = 2b + 4c = 0 \end{cases}$$

$$\begin{cases} 2a + 2c + 1 = 0 \\ 4b - 2c - 3 = 0 \\ b = -2c \end{cases} \rightarrow \begin{cases} a = -0.2 \\ b = 0.6 \\ c = -0.3 \end{cases}$$

Answer: $(\mathbf{-0.2; 0.6; -0.3})$.