Answer on Question #46072 – Math – Analytic Geometry

Question. Find the angle between the lines

$$P: \ x = 1, \qquad z - y = 0$$

and

$$Q: \ 2x - y = -1, \qquad z = 1.$$

Solution. Let us find vectors p and q which are parallel to the lines P and Q.

By assumption the line p is the intersection of two planes $x = 1$ and $z - y = 0$ having the following normal vectors:

$$a_1 = (1, 0, 0), \qquad a_2 = (0, -1, 1).$$

Hence p must be orthogonal to both a_1 and a_2. Therefore we can take p to be the cross-product $a_1 \times a_2$ of these vectors:

$$\begin{aligned} p = a_1 \times a_2 &= (1,0,0) \times (0,-1,1) \\ &= \left(\begin{vmatrix} 0 & 0 \\ -1 & 1 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} \right) \\ &= (0 \cdot 1 - 0 \cdot (-1), \ 0 \cdot 0 - 1 \cdot 1, \ 1 \cdot (-1) - 0 \cdot 0) \\ &= (0, -1 - 1)\,. \end{aligned}$$

Analogously, the line q is the intersection of two planes $2x - y = -1$ and $z = 1$ having normal vectors:

$$b_1 = (2, -1, 0), \qquad b_2 = (0, 0, 1)$$

and we can assume that

$$\begin{aligned} q = b_1 \times b_2 &= (2,-1,0) \times (0,0,1) \\ &= \left(\begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix}, \begin{vmatrix} 2 & -1 \\ 0 & 0 \end{vmatrix} \right) \\ &= (-1 \cdot 1 - 0 \cdot 0, \ 0 \cdot 0 - 1 \cdot 2, \ 2 \cdot 0 - 0 \cdot (-1)) \\ &= (-1, -2, 0). \end{aligned}$$

To find the angle ϕ between the vectors p and q notice that the scalar product (p, q) of these vectors can be computed in two distinct ways. On the one hand, it is the sum of products of the corresponding coordinates:

$$(p, q) = 0 \cdot (-1) + (-1) \cdot (-2) + (-1) \cdot 0 = 2.$$

On the other hand,

$$(p, q) = |p| \cdot |q| \cdot \cos\phi,$$

whence

$$\cos\phi = \frac{(p, q)}{|p| \cdot |q|}.$$

The lengths of these vectors are

$$|p| = \sqrt{0^2 + (-1)^2 + (-1)^2} = \sqrt{2}, \qquad |q| = \sqrt{(-1)^2 + (-2)^2 + 0^2} = \sqrt{5},$$

therefore

$$\cos\phi = \frac{(p, q)}{|p| \cdot |q|} = \frac{2}{\sqrt{2} \cdot \sqrt{5}} = \frac{\sqrt{2}}{\sqrt{5}} = \sqrt{\frac{2}{5}} = \sqrt{0.4} \approx 0.6325.$$

Hence

$$\phi = \arccos\sqrt{0.4} \approx \arccos(0.6325) \approx 50.77°.$$

Answer. The angle between the lines is

$$\arccos\sqrt{0.4} \approx 50.77°.$$