Answer on Question #46067 – Math - Linear Algebra

Problem.

Let T : R^2 -> R^2 and S: R^2 -> R^2 be linear operators defined by
T (x(subscript1) , x(subscript2)) = (x(subscript1) + x(subscript2) , x(subscript1) - x(subscript2)) and
S(x(subscript1) , x(subscript2)) = (x(subscript1) , x(subscript1) + 2x(subscript2))
respectively.
i) Find ToS and SoT.
ii) Let B ={ (1;0) , (0;1) } be the standard basis of R^3. Verify that
[ToS](subscriptB) = [T](subscriptB) o [S](subscriptB).

Solution:

$T\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}x_1+x_2\\x_1-x_2\end{pmatrix}$ and $S\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}x_1\\x_1+2x_2\end{pmatrix}$, so $T = \begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}$ and $S = \begin{pmatrix}1 & 1\\0 & 2\end{pmatrix}$.

i) $TS = \begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}1 & 1\\0 & 2\end{pmatrix} = \begin{pmatrix}1 & 3\\1 & -1\end{pmatrix}$ and $ST = \begin{pmatrix}1 & 1\\0 & 2\end{pmatrix}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix} = \begin{pmatrix}2 & 0\\2 & -2\end{pmatrix}$.

ii) $S\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1 & 1\\0 & 2\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$, $T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$, so $T(S)\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$.

$TS\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1 & 3\\1 & -1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} = T(S)\begin{pmatrix}1\\0\end{pmatrix}$.

$S\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1 & 1\\0 & 2\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$, $T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}3\\-1\end{pmatrix}$, so $T(S)\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}3\\-1\end{pmatrix}$

$TS\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1 & 3\\1 & -1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}3\\-1\end{pmatrix} = T(S)\begin{pmatrix}0\\1\end{pmatrix}$.

Hence $[TS]_B = [T]_B[S]_B$, as the left and the right operators transform basis to the same vectors.