Answer on Question #45969 – Math – Integral Calculus

1. Find $\int \sec 3x \tan x \, dx$.

Solution.

First of all we will rewrite our integral:

$$\int \sec 3x \tan x \, dx = \int \frac{\tan x}{\cos 3x} dx = \int \frac{\sin x}{\cos 3x \cos x} dx.$$

Now we can use formula: $\cos 3x = 4\cos^3 x - 3\cos x$. Then we will use substitution:

$t = \cos^2 x$,

$x = \arccos \sqrt{t}$,

$dx = -\frac{dt}{2\sqrt{t}\sqrt{1-t}}$.

After this we will have:

$$\int \frac{\sin x}{\cos 3x \cos x} dx = \int \frac{\sin x}{(4\cos^3 x - 3\cos x)\cos x} dx = \int \frac{\sqrt{1-\cos^2 x}}{4\cos^4 x - 3\cos^2 x} dx =$$

$$= -\frac{1}{2}\int \frac{\sqrt{1-t}}{\sqrt{t}(4t^2-3t)\sqrt{1-t}} dt = -\frac{1}{2}\int \frac{1}{t^{\frac{3}{2}}(4t-3)} dt = -\frac{1}{2}\int t^{-\frac{3}{2}}(4t-3)^{-1} dt.$$

We obtained the Chebyshev integral:

$$\int x^m (a + bx^n)^p dx.$$

When p is integer, as in our case, we can do substitution $x = t^s$, where s is least common denominator of m and n.

In our case $m = -\frac{3}{2}$ and $n = 1$. Hence, least common denominator is 2. And now we can do substitution $t = y^2$and $dt = 2ydy$:

$$-\frac{1}{2}\int t^{-\frac{3}{2}}(4t-3)^{-1} dt = -\frac{1}{2}\int y^{-3}(4y^2-3)^{-1} 2y dy = -\int \frac{1}{y^2(4y^2-3)} dy =$$

$$= \frac{1}{3}\int \frac{1}{y^2(1-\frac{4}{3}y^2)} dx.$$

We can rewrite our fraction:

$$\frac{1}{y^2(1-\frac{4}{3}y^2)} = \frac{1}{y^2} + \frac{\frac{4}{3}}{1-\frac{4}{3}y^2}.$$

Hence, we will have two integrals:

$$\frac{1}{3}\int \frac{1}{y^2} dy + \frac{4}{9}\int \frac{1}{1-\frac{4}{3}y^2} dy = -\frac{1}{3y} + \frac{4}{9}\sqrt{\frac{3}{4}} \operatorname{arctanh} \sqrt{\frac{4}{3}} y.$$

Now we must return to x:

$$-\frac{1}{3y}+\frac{4}{9}\sqrt{\frac{3}{4}}\operatorname{arctanh}\sqrt{\frac{4}{3}}y=-\frac{1}{3\sqrt{t}}+\frac{2}{3\sqrt{3}}\operatorname{arctanh}\frac{2}{\sqrt{3}}\sqrt{t}=-\frac{1}{3\cos x}+$$

$$+\frac{2}{3\sqrt{3}}\operatorname{arctanh}(\frac{2}{\sqrt{3}}\cos x).$$

Answer:

result is $-\frac{1}{3\cos x}+\frac{2}{3\sqrt{3}}\operatorname{arctanh}(\frac{2}{\sqrt{3}}\cos x)$.