Answer on Question #45940 – Math – Statistics and Probability

A fair coin is tossed until for the first time the same result appears twice in succession (i.e until the TT or HH). Describe the sample space and assign probabilities that reflects the fairness of the coin. Find the probabilities of the events

$A =$ "$The\ coin\ is\ tossed\ at\ most\ five\ times$", $B =$ "$the\ total\ number\ of\ tossed\ is\ odd'$", $C = AUB$

$D = A(intersection)B.$

Solution

Suppose that we obtain same results at k-th and k + 1-th coin toss.

If we obtain TT, then at k − 1-th coin toss we had received H (as we obtain two same results for the first time), k − 2-th coin toss we had received T, k − 3-th coin toss we had received H, k − 2-th coin toss we had received, etc. The probability of such event is $\left(\frac{1}{2}\right)^{k+1}$.

If we obtain HH, then at k − 1-th coin toss we had received T (as we obtain two same results for the first time), k − 2-th coin toss we had received H, k − 3-th coin toss we had received T, k − 2-th coin toss we had received, etc. The probability of such event is $\left(\frac{1}{2}\right)^{k+1}$.

The probability to obtain same results at k-th and k + 1-th coin toss is

$$\left(\frac{1}{2}\right)^{k+1} + \left(\frac{1}{2}\right)^{k+1} = \left(\frac{1}{2}\right)^{k}.$$

The probability to obtain two same results for less than five coin toss is

$$\left(\frac{1}{2}\right)^{1} + \left(\frac{1}{2}\right)^{2} + \left(\frac{1}{2}\right)^{3} + \left(\frac{1}{2}\right)^{4} = 0.9375,$$

as the sum of probabilities to obtain two same results for 2, 3, 4 and 5 coin toss.

Hence $P(A) \ = \ 0.9375.$

The probability to obtain two same results for odd number of coin toss is

$$\left(\frac{1}{2}\right)^{2} + \left(\frac{1}{2}\right)^{4} + \left(\frac{1}{2}\right)^{6} + \cdots = \frac{\frac{1}{4}}{1 - \left(\frac{1}{2}\right)^{2}} = \frac{1}{3} \approx 0.3333,$$

as the sum of probabilities to obtain two same results for 3, 5, 7, 9, … coin toss.

Hence $P(B) \ = \frac{1}{3} \approx \ 0.3333.$

The probability to obtain two same results for less than five coin toss or for odd number of coin toss is

$$\frac{1}{2} + \left(\frac{1}{2}\right)^{3} + \left(\frac{1}{2}\right)^{2} + \left(\frac{1}{2}\right)^{4} + \left(\frac{1}{2}\right)^{6} + \cdots = \frac{1}{2} + \left(\frac{1}{2}\right)^{3} + \frac{\frac{1}{4}}{1 - \left(\frac{1}{2}\right)^{2}} = \frac{23}{24} \approx 0.9583,$$

as the sum of probabilities to obtain two same results for 2, 4 and for 3, 5, 7, 9, … coin toss.

Hence $P(C) = \frac{23}{24} \approx 0.9583.$

The probability to obtain two same results for less than five coin toss and for odd number of coin toss is

$$\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^4 = 0.3125,$$

as the sum of probabilities to obtain two same results for 3, 5 coin toss.

Hence $P(D) = 0.3125.$

Answer: $\boldsymbol{P(A) = 0.9375, P(B) = \frac{1}{3} \approx 0.3333, P(C) = \frac{23}{24} \approx 0.9583, P(D) = 0.3125}$.