Answer on Question #45870 – Math – Vector Calculus

Problem.

1.) For the scalar potential function ϕ = (x^2 + y^2 + z^2)^2 and the velocity vector field u=(y^2, z, x^2) calculate the following vector quantities:

a) ∇ϕ ; ∇· u

b) (∇^2)ϕ = (∇· ∇)ϕ ; (∇^2)u

c) ∇ × u

where u is a vector, and the vector operator ∇ = (∂/∂x , ∂/∂y, ∂/∂z)

Solution.

By u_x, u_y, u_z we will denote coordinates of vector field u.

a) $\nabla\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right) = (4x(x^2+y^2+z^2), 4y(x^2+y^2+z^2), 4z(x^2+y^2+z^2))$ by definition of del operator (or nabla operator).

$\nabla \cdot \vec{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = \frac{\partial}{\partial x}(y^2) + \frac{\partial}{\partial y}(z) + \frac{\partial}{\partial z}(x^2) = 0 + 0 + 0 = 0$ by definition of inner product of del operator (or nabla operator) and vector field.

b) $(\nabla^2)\phi = \nabla(\nabla\phi) = \nabla\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right) = \Delta\,\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = (4(x^2+y^2+z^2) + 8x^2) + (4(x^2+y^2+z^2) + 8y^2) + (4(x^2+y^2+z^2) + 8z^2) = 20(x^2+y^2+z^2)$ by definition of scalar Laplacian.

$(\nabla^2)\vec{u} = (\nabla^2 u_x, \nabla^2 u_y, \nabla^2 u_z) = (2,0,2)$, by definition of vector Laplasian.

c) $\nabla\times\vec{u} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u_x & u_y & u_z \end{vmatrix} = \left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}, \frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}, \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) = \left(\frac{\partial}{\partial y}(x^2) - \frac{\partial}{\partial z}(z), \frac{\partial}{\partial z}(y^2) - \frac{\partial}{\partial x}(x^2), \frac{\partial}{\partial x}(z) - \frac{\partial}{\partial y}(y^2)\right) = (-1, -2x, -2y)$, by definition of vector product of del operator (or nabla operator) and vector field.

Answer:

a) $\nabla\phi = (4x(x^2+y^2+z^2), 4y(x^2+y^2+z^2), 4z(x^2+y^2+z^2))$, $\nabla\cdot\vec{u} = 0$;

b) $(\nabla^2)\phi = 20(x^2+y^2+z^2)$, $(\nabla^2)\vec{u} = (2,0,2)$

c) $\nabla\times\vec{u} = (-1, -2x, -2y)$.