Answer on Question #45754 – Math - Algebra

a) If the sum of the roots of a cubic equation is 3, the sum of the squares of the roots is 11 and the sum of the cubes of the roots is 27, find the equation. Also, solve the equation and find all its roots.

b) Solve the equation $x^3 + 2x^2 - 2x - 1 = 0$ using Cardano's method.

c) Write the systems obtained in 6(a) and 6(b) in matrix form.

Solution

a) We have a cubic equation

$$x^3 + ax^2 + bx + c = 0.$$

Let the roots be denoted α, β, γ.

We know that $\alpha + \beta + \gamma = 3;\ \alpha^2 + \beta^2 + \gamma^2 = 11;\ \alpha^3 + \beta^3 + \gamma^3 = 27.$

then by Vieta's formula $\alpha + \beta + \gamma = -a = 3, (\alpha\beta + \beta\gamma + \gamma\alpha) = b,\ \ \alpha\beta\gamma = c.$

$$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha) \rightarrow$$

$$\rightarrow 2b = 2(\alpha\beta + \beta\gamma + \gamma\alpha) = (\alpha + \beta + \gamma)^2 - (\alpha^2 + \beta^2 + \gamma^2) = 3^2 - 11 = -2.$$

Then

$$a = -3, b = -1.$$

Substituting α, β, γ in the original equation and adding:

$$(\alpha^3 + \beta^3 + \gamma^3) - 3(\alpha^2 + \beta^2 + \gamma^2) - (\alpha + \beta + \gamma) + 3c = 0 \rightarrow 27 - 3 \cdot 11 - 3 + 3c = 0 \rightarrow c = 3.$$

So the original equation is

$$x^3 - 3x^2 - x + 3 = 0.$$

$$x^2(x - 3) - (x - 3) = (x - 3)(x^2 - 1) = (x - 3)(x - 1)(x + 1) = 0.$$

Since the roots of the equation $x^3 - 3x^2 - x + 3 = 0$ are -1; 1; 3.

Answer: $\boldsymbol{x^3 - 3x^2 - x + 3 = 0}$ **and -1; 1; 3.**

b) $x^3 + 2x^2 - 2x - 1 = 0.$

$$x = t - \frac{2}{3} \rightarrow t^3 + px + q = 0,$$

where $p = -2 - \frac{2^2}{3} = -\frac{10}{3}$ and $q = -1 + \frac{2 \cdot 2^3 - 9 \cdot 2(-2)}{27} = \frac{25}{27}.$

$$x_1 = \sqrt[3]{-\frac{25}{27 \cdot 2} + \sqrt{\left(\frac{25}{27 \cdot 2}\right)^2 + \frac{\left(-\frac{10}{3}\right)^3}{27}}} + \sqrt[3]{-\frac{25}{27 \cdot 2} - \sqrt{\left(\frac{25}{27 \cdot 2}\right)^2 + \frac{\left(-\frac{10}{3}\right)^3}{27}}}.$$

In this case $\left(\frac{25}{27 \cdot 2}\right)^2 + \frac{\left(-\frac{10}{3}\right)^3}{27} < 0$. This entails finding the cube roots of complex numbers.

That's why the Cardano's method cannot be applied to this problem.

But it is easy to see that $x = 1$ is the root of equation. Then

$$x^3 + 2x^2 - 2x - 1 = (x-1)(x^2 + 3x + 1) = 0.$$

We have

$$(x^2 + 3x + 1) = 0 \to x_{1,2} = \frac{-3 \pm \sqrt{3^2 - 4}}{2}.$$

The roots of equation are 1; $\frac{-3+\sqrt{5}}{2}$; $\frac{-3-\sqrt{5}}{2}$.

c) 6(a):

$$\begin{cases} x - 3y + 4z = 9 \\ 4x + 3y + 2z = 7 \\ y - 2x = 5 - 10z \end{cases} \to \begin{cases} x - 3y + 4z = 9 \\ 4x + 3y + 2z = 7 \\ -2x + y + 10z = 5 \end{cases}.$$

The matrix form is

$$\begin{pmatrix} 1 & -3 & 4 \\ 4 & 3 & 2 \\ -2 & 1 & 10 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ 7 \\ 5 \end{pmatrix}.$$

6(b):

$$\begin{cases} 12x + 8y = 440 \\ 5x + 3y = 175 \end{cases}.$$

The matrix form is

$$\begin{pmatrix} 12 & 8 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 440 \\ 175 \end{pmatrix}.$$