

Answer on Question #45752 – Math – Algebra

Let m, n, x, y, z be positive real numbers, with $x + y + z = 1$. Prove that

$$\frac{x^4}{(mx+ny)(my+nx)} + \frac{y^4}{(my+nz)(mz+ny)} + \frac{z^4}{(mz+nx)(mx+nz)} \text{ is greater than or equal to } \frac{1}{3(m+n)^2}.$$

[Hint: Apply the AM $\geq$ GM inequality to each term in the LHS, and then apply the CS inequality.]

Solution

$$\frac{(a+b)^2}{4} = \left(\frac{a+b}{2}\right)^2 \geq \left(\frac{2\sqrt{ab}}{2}\right)^2 = ab,$$

by AM-GM inequality.

Hence

$$\frac{(m+n)^2(x+y)^2}{4} = \left(\frac{(mx+ny) + (my+nx)}{2}\right)^2 \geq (mx+ny)(my+nx),$$

$$\frac{(m+n)^2(y+z)^2}{4} = \left(\frac{(my+nz) + (mz+ny)}{2}\right)^2 \geq (my+nz)(mz+ny),$$

$$\frac{(m+n)^2(x+z)^2}{4} = \left(\frac{(mx+nz) + (mz+nx)}{2}\right)^2 \geq (mx+nz)(mz+nx).$$

Therefore

$$\begin{aligned} & \frac{x^4}{(mx+ny)(my+nx)} + \frac{y^4}{(my+nz)(mz+ny)} + \frac{z^4}{(mz+nx)(mx+nz)} \\ & \geq \frac{4x^4}{(m+n)^2(x+y)^2} + \frac{4y^4}{(m+n)^2(y+z)^2} + \frac{4z^4}{(m+n)^2(x+z)^2} \\ & = \frac{4}{(m+n)^2} \left(\frac{x^4}{(x+y)^2} + \frac{y^4}{(y+z)^2} + \frac{z^4}{(x+z)^2} \right). \end{aligned}$$

$$\left(\frac{x^4}{(x+y)^2} + \frac{y^4}{(y+z)^2} + \frac{z^4}{(x+z)^2} \right) (1^2 + 1^2 + 1^2) \geq \left(\frac{x^2}{(x+y)} + \frac{y^2}{(y+z)} + \frac{z^2}{(x+z)} \right)^2$$

by Cauchy-Shwartz inequality, so

$$\frac{4}{(m+n)^2} \left(\frac{x^4}{(x+y)^2} + \frac{y^4}{(y+z)^2} + \frac{z^4}{(x+z)^2} \right) \geq \frac{4}{3(m+n)^2} \left(\frac{x^2}{(x+y)} + \frac{y^2}{(y+z)} + \frac{z^2}{(x+z)} \right)^2$$

and

$$\begin{aligned} & \frac{x^4}{(mx+ny)(my+nx)} + \frac{y^4}{(my+nz)(mz+ny)} + \frac{z^4}{(mz+nx)(mx+nz)} \\ & \geq \frac{4}{3(m+n)^2} \left(\frac{x^2}{(x+y)} + \frac{y^2}{(y+z)} + \frac{z^2}{(x+z)} \right)^2. \end{aligned}$$

$$\left(\frac{x^2}{(x+y)} + \frac{y^2}{(y+z)} + \frac{z^2}{(x+z)} \right) ((x+y) + (y+z) + (x+z)) \geq (x+y+z)^2$$

by Cauchy-Schwartz inequality, so

$$\begin{aligned}\frac{4}{3(m+n)^2} \left(\frac{x^2}{(x+y)} + \frac{y^2}{(y+z)} + \frac{z^2}{(x+z)} \right)^2 &\geq \frac{4}{3(m+n)^2} \frac{(x+y+z)^4}{((x+y) + (y+z) + (x+z))^2} \\ &= \frac{4}{3(m+n)^2} \frac{(1)^4}{(2)^2} = \frac{1}{3(m+n)^2}\end{aligned}$$

as $x + y + z = 1$.

Hence

$$\frac{x^4}{(mx+ny)(my+nx)} + \frac{y^4}{(my+nz)(mz+ny)} + \frac{z^4}{(mz+nx)(mx+nz)} \geq \frac{1}{3(m+n)^2}.$$