

Answer on Question #45629 – Math - Algebra

Problem.

Let m, n, x, y, z be positive real numbers, with $x + y + z = 1$. Prove that $(x^4)/(mx+ny)(my+nx) + (y^4)/(my+nz)(mz+ny) + (z^4)/(mz+nx)(mx+nz)$ is greater than or equal to $1/3(m+n)^2$

[Hint: Apply the $AM \geq GM$ inequality to each term in the LHS, and then apply the CS inequality.]

Solution.

$$\frac{(a+b)^2}{4} = \left(\frac{a+b}{2}\right)^2 \geq \left(\frac{2\sqrt{ab}}{2}\right)^2 \geq ab,$$

by AM-GM inequality.

Hence

$$\begin{aligned}\frac{(m+n)^2(x+y)^2}{4} &= \left(\frac{(mx+ny) + (my+nx)}{2}\right)^2 \geq (mx+ny)(my+nx), \\ \frac{(m+n)^2(y+z)^2}{4} &= \left(\frac{(my+nz) + (mz+ny)}{2}\right)^2 \geq (my+nz)(mz+ny), \\ \frac{(m+n)^2(x+z)^2}{4} &= \left(\frac{(mz+nx) + (mx+nz)}{2}\right)^2 \geq (mz+nx)(mx+nz),\end{aligned}$$

Therefore

$$\begin{aligned}\frac{x^4}{(mx+ny)(my+nx)} + \frac{y^4}{(my+nz)(mz+ny)} + \frac{z^4}{(mz+nx)(mx+nz)} \\ \geq \frac{4x^4}{(m+n)^2(x+y)^2} + \frac{4y^4}{(m+n)^2(y+z)^2} + \frac{4z^4}{(m+n)^2(x+z)^2} \\ = \frac{4}{(m+n)^2} \left(\frac{x^4}{(x+y)^2} + \frac{y^4}{(y+z)^2} + \frac{z^4}{(x+z)^2} \right) \\ \left(\frac{x^4}{(x+y)^2} + \frac{y^4}{(y+z)^2} + \frac{z^4}{(x+z)^2} \right) (1^2 + 1^2 + 1^2) \geq \left(\frac{x^2}{x+y} + \frac{y^2}{y+z} + \frac{z^2}{x+z} \right)^2\end{aligned}$$

by Cauchy-Schwartz inequality, so

$$\frac{4}{(m+n)^2} \left(\frac{x^4}{(x+y)^2} + \frac{y^4}{(y+z)^2} + \frac{z^4}{(x+z)^2} \right) \geq \frac{4}{3(m+n)^2} \left(\frac{x^2}{x+y} + \frac{y^2}{y+z} + \frac{z^2}{x+z} \right)^2$$

and

$$\begin{aligned}\frac{x^4}{(mx+ny)(my+nx)} + \frac{y^4}{(my+nz)(mz+ny)} + \frac{z^4}{(mz+nx)(mx+nz)} \\ \geq \frac{4}{3(m+n)^2} \left(\frac{x^2}{x+y} + \frac{y^2}{y+z} + \frac{z^2}{x+z} \right)^2 \\ \left(\frac{x^2}{x+y} + \frac{y^2}{y+z} + \frac{z^2}{x+z} \right) ((x+y) + (y+z) + (x+z)) \geq (x+y+z)^2\end{aligned}$$

by Cauchy-Schwartz inequality, so

$$\frac{4}{3(m+n)^2} \left(\frac{x^2}{x+y} + \frac{y^2}{y+z} + \frac{z^2}{x+z} \right)^2 \geq \frac{1}{3(m+n)^2},$$

as $x + y + z = 1$.

Hence

$$\frac{x^4}{(mx+ny)(my+nx)} + \frac{y^4}{(my+nz)(mz+ny)} + \frac{z^4}{(mz+nx)(mx+nz)} \geq \frac{1}{3(m+n)^2}.$$