Answer on Question #45563 – Math - Abstract Algebra

Problem.

a) Let
S =

a 0
0 b

a; b 2 Z

:
i) Check that S is a subring of M2
(R) and it is a commutative ring with identity.
ii) Is S an ideal of M2
(R)? Justify your answer.
iii) Is S an integral domain? Justify your answer.
iv) Find all the units of the ring S.
v) Check whether
I =
a 0
0 b
a; b 2 Z; 2 j a
:
is an ideal of S.
vi) Show that S ' Z Z where the addition and multiplication operations are componentwise addition and multiplication.
b) Let G = S
4, H = A4
and K = f1; (1 2)(3 4); (1 3)(2 4); (1 4)(2 3)g.
i) Check that H=K = h(1 2 3)Hi
ii) Check that K is normal in H.(Hint: For each h 2 H,h 62 K, check that hK = Kh.)
iii) Check whether (1 2 3 4))H is the inverse of (1 3 4 2)H in the group S
4
=H.

Remark.

The statement isn't correctly formatted. I suppose that the correct statement is
"**a)** Let
$S = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \middle| a.b \in \mathbb{Z} \right\}$.
i) Check that S is a subring of $M_2(\mathbb{R})$ and it is a commutative ring with identity.
ii) Is S an ideal of $M_2(\mathbb{R})$? Justify your answer.
iii) Is S an integral domain? Justify your answer.
iv) Find all units of the ring S.
v) Check whether

$$I = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \middle| a.b \in \mathbb{Z}, 2|a \right\}$$

is an ideal of S.
vi) Show that $S \simeq \mathbb{Z} \times \mathbb{Z}$ where the addition and multiplication are component component wise addition and multiplication.

b) Let $G = S_4$, $H = A_4$ and $K = \{1;\ (1\ 2)(3\ 4);\ (1\ 3)(2\ 4);\ (1\ 4)(2\ 3)\}$.
i) Check that $H/K = \langle (1\ 2\ 3)H \rangle$.
ii) Check that K is normal in H.(Hint: For each $h \in H, h \notin K$, check that $hK = Kh$.)
iii) Check whether $(1\ 2\ 3\ 4)H$ is the inverse of $(1\ 3\ 4\ 2)H$ in the group S_4/H."

Solution.

a)

i)
(1) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in S$, so S consists multiplicative identity;
(2) For all $\begin{bmatrix} a_1 & 0 \\ 0 & b_1 \end{bmatrix}, \begin{bmatrix} a_2 & 0 \\ 0 & b_2 \end{bmatrix} \in S$ $\begin{bmatrix} a_1 & 0 \\ 0 & b_1 \end{bmatrix} + \begin{bmatrix} -a_2 & 0 \\ 0 & -b_2 \end{bmatrix} = \begin{bmatrix} -a_2 & 0 \\ 0 & -b_2 \end{bmatrix} + \begin{bmatrix} a_1 & 0 \\ 0 & b_1 \end{bmatrix} = \begin{bmatrix} a_1 - a_2 & 0 \\ 0 & b_1 - b_2 \end{bmatrix} \in S$, so S is closed under subtractions and S is commutative group over addition.
(3) For all $\begin{bmatrix} a_1 & 0 \\ 0 & b_1 \end{bmatrix}, \begin{bmatrix} a_2 & 0 \\ 0 & b_2 \end{bmatrix} \in S$ $\begin{bmatrix} a_1 & 0 \\ 0 & b_1 \end{bmatrix}\begin{bmatrix} a_2 & 0 \\ 0 & b_2 \end{bmatrix} = \begin{bmatrix} a_1a_2 & 0 \\ 0 & b_1b_2 \end{bmatrix} \in S$, so S is closed under multiplications.
From (1)-(3) and subring test S is commutative subring with unity.
ii) For $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in S$ and for $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \in M_2(\mathbb{R})$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \notin S$, so S isn't a ideal.
iii) For $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in S$ and for $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in S$ $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, so S has divisor of zero and isn't integral domain.
iv) Suppose that $\begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix}$ is unit of S, then for all $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in S$

$$\begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}.$$

$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} e_1a & 0 \\ 0 & e_2b \end{bmatrix}$. Then $e_1 = e_2 = 1$. Hence $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is unique unit.
v) For $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \in I$ and for $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \in M_2(\mathbb{R})$ $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \notin I$, so I isn't an ideal.
vi) Let $: S \to (\mathbb{Z}, \mathbb{Z})$ $f\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\right) = (a, b)$. For all $\begin{bmatrix} a_1 & 0 \\ 0 & b_1 \end{bmatrix}, \begin{bmatrix} a_2 & 0 \\ 0 & b_2 \end{bmatrix} \in S$

$$f\left(\begin{bmatrix} a_1 & 0 \\ 0 & b_1 \end{bmatrix}\right) + f\left(\begin{bmatrix} a_2 & 0 \\ 0 & b_2 \end{bmatrix}\right) = (a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$$
$$= f\left(\begin{bmatrix} a_1 + a_2 & 0 \\ 0 & b_1 + b_2 \end{bmatrix}\right) = (a_1 + a_2, b_1 + b_2),$$

$f\left(\begin{bmatrix} a_1 & 0 \\ 0 & b_1 \end{bmatrix}\right) f\left(\begin{bmatrix} a_2 & 0 \\ 0 & b_2 \end{bmatrix}\right) = (a_1, b_1)(a_2, b_2) = (a_1a_2, b_1b_2) = f\left(\begin{bmatrix} a_1a_2 & 0 \\ 0 & b_1b_2 \end{bmatrix}\right) = (a_1a_2, b_1b_2)$
and

$$f\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = (1,1).$$

Hence f is ring homorphism. f is also one-to-one and onto map, so f is isomorphism and $S \backsimeq \mathbb{Z} \times \mathbb{Z}$.

b)

$$H = \left\{ \begin{matrix} 1; (1\ 2)(3\ 4);\ (1\ 3)(2\ 4);\ (1\ 4)(2\ 3); (1\ 2\ 3); (1\ 2\ 4);\ (1\ 3\ 2);\ (1\ 3\ 4);\ (1\ 4\ 2); \\ (1\ 4\ 3);\ (2\ 3\ 4);\ (2\ 4\ 3) \end{matrix} \right\}.$$

i) $(1\ 2\ 3)H =$

$$\left\{ \begin{matrix} (1\ 2\ 3); (1\ 2\ 3)(1\ 2)(3\ 4);\ (1\ 2\ 3)(1\ 3)(2\ 4);\ (1\ 2\ 3)(1\ 4)(2\ 3); \\ (1\ 2\ 3)(1\ 2\ 3); (1\ 2\ 3)(1\ 2\ 4);\ (1\ 2\ 3)(1\ 3\ 2);\ (1\ 2\ 3)(1\ 3\ 4);\ (1\ 2\ 3)(1\ 4\ 2); \\ (1\ 2\ 3)(1\ 4\ 3); (1\ 2\ 3)(2\ 3\ 4);\ (1\ 2\ 3)(2\ 4\ 3) \end{matrix} \right\} =$$

$\{(1\ 2\ 3); (1\ 3\ 4); (2\ 4\ 3); (1\ 4\ 2); (1\ 3\ 2); \dots\}$
There are no class in H/K that has 5 different elements (it has at most 4 elements).

ii) We will check that for all$h \in H$, $h \notin K$, check that $hK = Kh$:

- $(1\,2\,3)H = \{(1\,2\,3);\ (1\,2\,3)(1\,2)(3\,4);\ (1\,2\,3)(1\,3)(2\,4);\ (1\,2\,3)(1\,4)(2\,3)\} = \{(1\,2\,3); (1\,3\,4); (2\,4\,3); (1\,4\,2)\}$ and
  $H(1\,2\,3) = \{(1\,2\,3);\ (1\,2)(3\,4)(1\,2\,3);\ (1\,3)(2\,4)(1\,2\,3);\ (1\,4)(2\,3)(1\,2\,3)\} = \{(1\,2\,3); (2\,4\,3); (1\,4\,2); (1\,3\,4)\}$.
  Hence $(1\,2\,3)H = H(1\,2\,3)$.
- $(1\,2\,4)H = \{(1\,2\,4);\ (1\,2\,4)(1\,2)(3\,4);\ (1\,2\,4)(1\,3)(2\,4);\ (1\,2\,4)(1\,4)(2\,3)\} = \{(1\,2\,4); (1\,4\,3); (1\,3\,2); (2\,3\,4)\}$ and
  $H(1\,2\,4) = \{(1\,2\,4);\ (1\,2)(3\,4)(1\,2\,4);\ (1\,3)(2\,4)(1\,2\,4);\ (1\,4)(2\,3)(1\,2\,4)\} = \{(1\,2\,4); (2\,3\,4); (1\,4\,3); (1\,3\,2)\}$.
  Hence $(1\,2\,4)H = H(1\,2\,4)$.
- $(1\,3\,2)H = \{(1\,3\,2);\ (1\,3\,2)(1\,2)(3\,4);\ (1\,3\,2)(1\,3)(2\,4);\ (1\,3\,2)(1\,4)(2\,3)\} = \{(1\,3\,2); (2\,3\,4); (1\,2\,4); (1\,4\,3)\}$ and
  $H(1\,3\,2) = \{(1\,3\,2);\ (1\,2)(3\,4)(1\,3\,2);\ (1\,3)(2\,4)(1\,3\,2);\ (1\,4)(2\,3)(1\,3\,2)\} = \{(1\,3\,2); (1\,4\,3); (2\,3\,4); (1\,2\,4)\}$.
  Hence $(1\,3\,2)H = H(1\,3\,2)$.
- $(1\,3\,4)H = \{(1\,3\,4);\ (1\,3\,4)(1\,2)(3\,4);\ (1\,3\,4)(1\,3)(2\,4);\ (1\,3\,4)(1\,4)(2\,3)\} = \{(1\,3\,4); (1\,2\,3); (1\,4\,2); (2\,4\,3)\}$ and
  $H(1\,3\,4) = \{(1\,3\,4);\ (1\,2)(3\,4)(1\,3\,4);\ (1\,3)(2\,4)(1\,3\,4);\ (1\,4)(2\,3)(1\,3\,4)\} = \{(1\,3\,4); (1\,4\,2); (2\,4\,3); (1\,2\,3)\}$.
  Hence $(1\,3\,4)H = H(1\,3\,4)$.
- $(1\,4\,2)H = \{(1\,4\,2);\ (1\,4\,2)(1\,2)(3\,4);\ (1\,4\,2)(1\,3)(2\,4);\ (1\,4\,2)(1\,4)(2\,3)\} = \{(1\,4\,2); (2\,4\,3); (1\,3\,4); (1\,2\,3)\}$ and
  $H(1\,4\,2) = \{(1\,4\,2);\ (1\,2)(3\,4)(1\,4\,2);\ (1\,3)(2\,4)(1\,4\,2);\ (1\,4)(2\,3)(1\,4\,2)\} = \{(1\,4\,2); (1\,3\,4); (1\,2\,3); (2\,4\,3)\}$.
  Hence $(1\,4\,2)H = H(1\,4\,2)$.
- $(1\,4\,3)H = \{(1\,4\,3);\ (1\,4\,3)(1\,2)(3\,4);\ (1\,4\,3)(1\,3)(2\,4);\ (1\,4\,3)(1\,4)(2\,3)\} = \{(1\,4\,3); (1\,2\,4); (2\,3\,4); (1\,3\,2)\}$ and
  $H(1\,4\,3) = \{(1\,4\,3);\ (1\,2)(3\,4)(1\,4\,3);\ (1\,3)(2\,4)(1\,4\,3);\ (1\,4)(2\,3)(1\,4\,3)\} = \{(1\,4\,3); (1\,3\,2); (1\,2\,4); (2\,3\,4)\}$.
  Hence $(1\,4\,3)H = H(1\,4\,3)$.
- $(2\,3\,4)H = \{(2\,3\,4);\ (2\,3\,4)(1\,2)(3\,4);\ (2\,3\,4)(1\,3)(2\,4);\ (2\,3\,4)(1\,4)(2\,3)\} = \{(2\,3\,4); (1\,3\,2); (1\,4\,3); (1\,2\,4)\}$ and
  $H(2\,3\,4) = \{(2\,3\,4);\ (1\,2)(3\,4)(2\,3\,4);\ (1\,3)(2\,4)(2\,3\,4);\ (1\,4)(2\,3)(2\,3\,4)\} = \{(2\,3\,4); (1\,2\,4); (1\,3\,2); (1\,4\,3)\}$.
  Hence $(2\,3\,4)H = H(2\,3\,4)$.
- $(2\,4\,3)H = \{(2\,4\,3);\ (2\,4\,3)(1\,2)(3\,4);\ (2\,4\,3)(1\,3)(2\,4);\ (2\,4\,3)(1\,4)(2\,3)\} = \{(2\,4\,3); (1\,4\,2); (1\,2\,3); (1\,3\,4)\}$ and
  $H(2\,4\,3) = \{(2\,4\,3);\ (1\,2)(3\,4)(2\,4\,3);\ (1\,3)(2\,4)(2\,4\,3);\ (1\,4)(2\,3)(2\,4\,3)\} = \{(2\,4\,3); (1\,2\,3); (1\,3\,4); (1\,4\,2)\}$.
  Hence $(2\,4\,3)H = H(2\,4\,3)$.

Therefore K is normal in H.

iii) $(1\,2\,3\,4)H(1\,3\,4\,2)H = (1\,2\,3\,4)(1\,3\,4\,2)H = (1\,4\,3)H \neq H$. Therefore $(1\,2\,3\,4)H$ isn't an inverse to $(1\,3\,4\,2)H$.