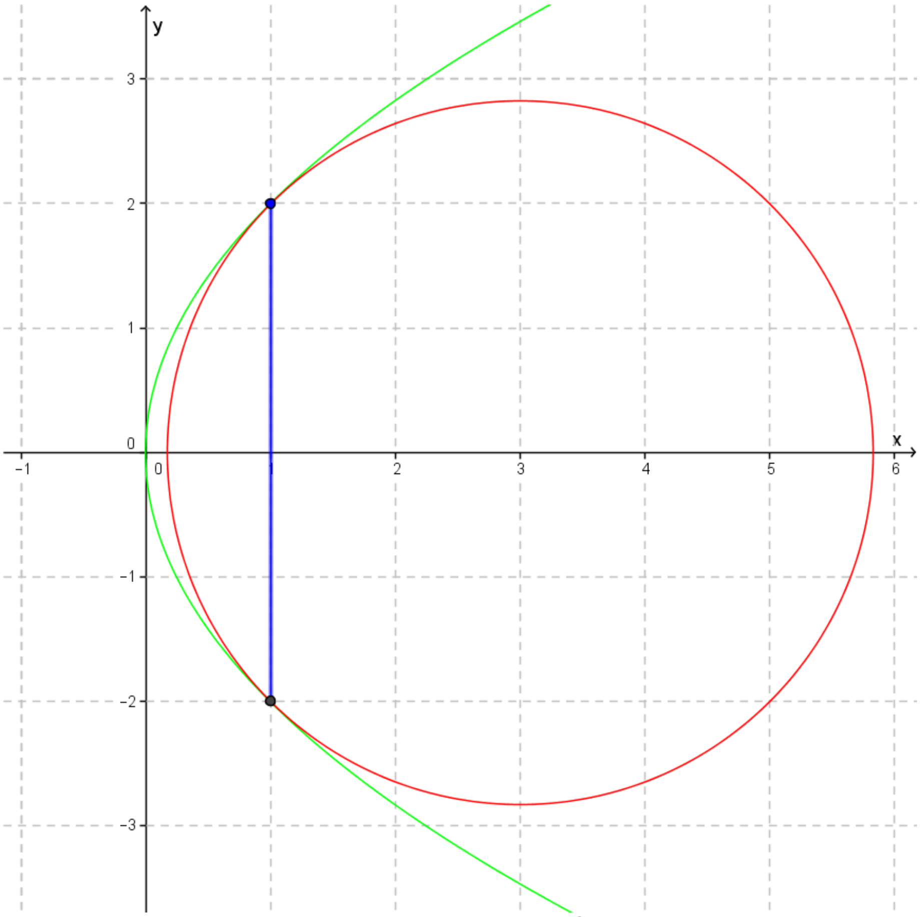
Answer on Question #45506 – Math - Analytic Geometry

Problem.

r a circle described on any focal chord of the parabola y2=4ax as its diameter will touch which part of parabola?

Solution.

Suppose that the center of the circle is $O(x_0, y_0)$. Then $y_0 = 0$, as the parabola symmetric about the x-axis. The equation of the circle with center $O(x_0, 0)$ and radius r is $(x - x_0)^2 + y^2 = r^2$.
The points of intersection of the circle $(x - x_0)^2 + y^2 = 1$ and the parabola $y^2 = 4ax$ are the solution of the equation $\begin{cases}(x - x_0)^2 + y^2 = r^2; \\ y^2 = 4ax,\end{cases}$ or $\begin{cases}(x - x_0)^2 + 4ax = r^2; \\ y^2 = 4ax.\end{cases}$ The circle touches the parabola if and only if the equation $(x - x_0)^2 + 4ax = r^2$ $(i.e.$ $x^2 - x(2x_0 - 4a) + x_0^2 - r^2 = 0)$ has only one positive solution or when $(0,0)$ is the point of their intersection.
The equation $x^2 - x(2x_0 - 4a) + x_0^2 - r^2 = 0$ has one positive solution when $D = (2x_0 - 4a)^2 - 4(x_0^2 - r^2) = 4x_0^2 - 16x_0a + 16a^2 - 4x_0^2 + 4r^2 = 0$ or $x_0 = a + \frac{r^2}{4a}$ and $x_0 - 2a = \frac{r^2}{4a} - a > 0$ $(r > 2a)$.
Therefore if $r > 2a$, then the equation of the circle with radius r that touches the parabola is $\left(x - a - \frac{r^2}{4a}\right)^2 + y^2 = r^2$ and if $r \le 2a$, then the equation of circle with radius r that touches the parabola is $(x - r)^2 + y^2 = r^2$.
The focal chord is the chord of the circle if $r > 2a$ and $\frac{r^2}{4a} - a = a$ (the solution of the equation $x^2 - x(2x_0 - 4a) + x_0^2 - r^2 = 0$ is equal to x-coordinate of focus). Hence $r = 2\sqrt{2}a$. This chord couldn't be diameter, as $x = a + \frac{r^2}{4a}$ isn't focal chord.
Answer: The chord of the circle $(x - 3a)^2 + y^2 = 8a^2$ is the focal chord of the parabola $y^2 = 4ax$ that touches this circle.
The picture when $a = 1$.



There no circle such that the focal chord of the parabola $y^2 = 4ax$ touches this circle and is the diameter of this circle.