Answer on Question #45473 – Math - Algebra

Problem.

Let m, n, x, y, z be positive real numbers, with x + y + z = 1. Prove that
(x^4)/(mx+ny)(my+nx) + (y^4)/(my+nz)(mz+ny) + (z^4)/(mz+nx)(mx+nz) is greater than or equal to 1/3(m+n)^2
[Hint: Apply the AM ≥ GM inequality to each term in the LHS, and then apply the CS inequality.]

Solution.

$$\frac{(a+b)^2}{4} = \left(\frac{a+b}{2}\right)^2 \ge \left(\frac{2\sqrt{ab}}{2}\right)^2 \ge ab,$$

by AM-GM inequality.
Hence

$$\frac{(m+n)^2(x+y)^2}{4} = \left(\frac{(mx+ny)+(my+nx)}{2}\right)^2 \ge (mx+ny)(my+nx),$$
$$\frac{(m+n)^2(y+z)^2}{4} = \left(\frac{(my+nz)+(mz+ny)}{2}\right)^2 \ge (my+nz)(mz+ny),$$
$$\frac{(m+n)^2(x+z)^2}{4} = \left(\frac{(mz+nx)+(mx+nz)}{2}\right)^2 \ge (mz+nx)(mx+nz),$$

Therefore

$$\frac{x^4}{(mx+ny)(my+nx)} + \frac{y^4}{(my+nz)(mz+ny)} + \frac{z^4}{(mz+nx)(mx+nz)}$$
$$\ge \frac{4x^4}{(m+n)^2(x+y)^2} + \frac{4y^4}{(m+n)^2(y+z)^2} + \frac{4z^4}{(m+n)^2(x+z)^2}$$
$$= \frac{4}{(m+n)^2}\left(\frac{x^4}{(x+y)^2} + \frac{y^4}{(y+z)^2} + \frac{z^4}{(x+z)^2}\right).$$

$$\left(\frac{x^4}{(x+y)^2} + \frac{y^4}{(y+z)^2} + \frac{z^4}{(x+z)^2}\right)(1^2+1^2+1^2) \ge \left(\frac{x^2}{x+y} + \frac{y^2}{y+z} + \frac{z^2}{x+z}\right)^2$$

by Cauchy-Shwartz inequality, so

$$\frac{4}{(m+n)^2}\left(\frac{x^4}{(x+y)^2} + \frac{y^4}{(y+z)^2} + \frac{z^4}{(x+z)^2}\right) \ge \frac{4}{3(m+n)^2}\left(\frac{x^2}{x+y} + \frac{y^2}{y+z} + \frac{z^2}{x+z}\right)^2$$

and

$$\frac{x^4}{(mx+ny)(my+nx)} + \frac{y^4}{(my+nz)(mz+ny)} + \frac{z^4}{(mz+nx)(mx+nz)}$$
$$\ge \frac{4}{3(m+n)^2}\left(\frac{x^2}{x+y} + \frac{y^2}{y+z} + \frac{z^2}{x+z}\right)^2.$$

$$\left(\frac{x^2}{x+y} + \frac{y^2}{y+z} + \frac{z^2}{x+z}\right)\big((x+y)+(y+z)+(x+z)\big) \ge (x+y+z)^2$$

by Cauchy-Shwartz inequality, so

$$\frac{4}{3(m+n)^2}\left(\frac{x^2}{x+y} + \frac{y^2}{y+z} + \frac{z^2}{x+z}\right)^2 \ge \frac{1}{3(m+n)^2},$$

as $x+y+z=1$.
Hence

$$\frac{x^4}{(mx+ny)(my+nx)} + \frac{y^4}{(my+nz)(mz+ny)} + \frac{z^4}{(mz+nx)(mx+nz)} \ge \frac{1}{3(m+n)^2}.$$