Answer on Question #45471 – Math - Abstract Algebra

Problem.

The map f : R[x] -> M(subscript3)(R) is defined by
f (a(subscript 0) + a(subscript 1)x + a(subscript 2)x(power 2) + ... + a(subscript n)x(power n)
_ _
| a(subscript 0) a(subscript 1) a(subscript 2) |
= | 0 a(subscript 0) a (subscript 1) |
|_ 0 0 a(subscript 0) _|
Show that f is a group homomorphism.Determine ker(f) also.

Solution.

Let $P(x) = a_0 + a_1 x + \cdots + a_n x^n$ and $Q(x) = b_0 + b_1 x + \cdots + b_n x^n$. Then

$$f(P(x))f(Q(x)) = \begin{bmatrix} a_0 & a_1 & a_2 \\ 0 & a_0 & a_1 \\ 0 & 0 & a_0 \end{bmatrix} \begin{bmatrix} b_0 & b_1 & b_2 \\ 0 & b_0 & b_1 \\ 0 & 0 & b_0 \end{bmatrix} = \begin{bmatrix} a_0 b_0 & a_0 b_1 + a_1 b_0 & a_0 b_2 + a_1 b_1 + a_2 b_0 \\ 0 & a_0 b_0 & a_0 b_1 + a_1 b_0 \\ 0 & 0 & a_0 b_0 \end{bmatrix}$$

and

$$\begin{aligned} f(P(x)Q(x)) &= f(a_0 b_0 + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + \cdots) \\ &= \begin{bmatrix} a_0 b_0 & a_0 b_1 + a_1 b_0 & a_0 b_2 + a_1 b_1 + a_2 b_0 \\ 0 & a_0 b_0 & a_0 b_1 + a_1 b_0 \\ 0 & 0 & a_0 b_0 \end{bmatrix}. \end{aligned}$$

Hence $f(P(x))f(Q(x)) = f(P(x)Q(x))$.
Then f is group homorphism.

$$\ker f = \left\{ P \in \mathbb{R}[x] \middle| f(P) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\} = \{a_0 + a_1 x + \cdots + a_n x^n \in \mathbb{R}[x] | a_0 = a_1 = a_2 = 0\}.$$

$$\ker f = \{a_3 x^3 + \cdots + a_n x^n | a_i \in \mathbb{R}, i = 3..n\}.$$

Answer: $\ker f = \{a_3 x^3 + \cdots + a_n x^n | a_i \in \mathbb{R}, i = 3..n\}$.