Answer on Question #45406 – Math - Abstract Algebra

Problem.

Let D(subscript12) = ({x,y : x^2 = e ; y^6 = e ; xy =(y^-1) x})
a) Which of the following subsets are subgroups of D(subscript12) ? Justify your answer.
i) {x,y,xy,y^2,y^3,e} ii) {xy,xy^2,y^2,e} iii) {x,y^3,xy^3,e}
b) Find the order of y^2. Is the subgroup (y^2) normal ? Justify your answer.
c) Let D(subscript2n) = ({x,y : x^2 = e ; y^n = e ; xy =(y^-1) x})
Prove the relation
{ x^i*y^(j+l) if k is even
X^i*y^j*x^k*y^l = {
{ x^(i+k)*y^(l-j) if k is odd
Further, find all the elements of order 2 in D(subscript12) .
d) Find two different Sylow 2-subgroups of D(subscript12) .

Solution.

a) G is supgroup of D_{12} if and only if it is closed under product and inverses.
i) Suppose $G_1 = \{x, y, xy, y^2, y^3, e\}$ is a subgroup of D_{12}. Then $y^2 \cdot y^3 = y^5 \in G_1$. Hence y^5 should be equal to either x or y or xy or y^2 or y^3.
If $x = y^5$, then $e = x^2 = y^{10} = y^4$ or $y^2 = e$, what is impossible (the order of y is 6).
If $y = y^5$, then $e = y^4$ or $y^2 = e$, what is impossible (the order of y is 6).
If $xy = y^5$, then $x = y^4$ or $e = x^2 = y^8 = y^3$, what is impossible (the order of y is 6).
If $y^2 = y^5$, then $e = y^3$, what is impossible (the order of y is 6).
If $y^3 = y^5$, then $e = y^2$, what is impossible (the order of y is 6).
Hence y^5 isn't equal to neither x nor y nor xy nor y^2 nor y^3 nor e. Therefore G_1 isn't a subgroup.
ii) Suppose $G_2 = \{xy, xy^2, y^2, e\}$ is a subgroup of D_{12}. Then $y^2 \cdot y^2 = y^4 \in G_2$. Hence y^4 should be equal to either xy or xy^2 or y^2.
If $xy = y^4$, then $x = y^3$. Then $G_2 = \{y^4, y^5, y^2, e\}$, what is impossible ($y^5$ doesn't have inverse).
If $xy^2 = y^4$, then $x = y^2$, $e = x^2 = y^4$ or $y^2 = e$, what is impossible (the order of y is 6).
If $y^2 = y^5$, then $e = y^3$, what is impossible (the order of y is 6).
Hence y^4 isn't equal to neither xy nor xy^2 nor y^2. Therefore G_2 isn't a subgroup.
iii) $G_3 = \{x, y^3, xy^3, e\}$.
$x^{-1} = x$, as $x^2 = e$;
$(y^3)^{-1} = y^3$, as $y^6 = e$;
$(xy^3)^{-1} = xy^3$, as $(xy^3)^2 = xy^3 \cdot y^3x = xy^6x = x^2 = e$ ($xy^3 = (y^3)^{-1}\ x = y^3x$);
$e^{-1} = e$;
Hence G_3 is closed under inverses.
$y^3 \cdot x = x \cdot y^3 \in G_3$;
$x \cdot xy^3 = x^2y^3 = y^3 \in G_3$;
$xy^3 \cdot x = y^3x \cdot x = y^3 \in G_3$ ($xy^3 = (y^3)^{-1}\ x = y^3x$);
$y^3 \cdot xy^3 = y^3 \cdot y^3x = y^6x = x \in G_3$;
$xy^3 \cdot y^3 = xy^6 = x \in G_3$;
Hence G_3 is closed under products.
Therefore G_3 is a subgroup.
Answer: iii).

b) $(y^2)^2 = y^4$ and $(y^2)^3 = e$, so the order of y^2 is 3. The subgroup (y^2) is cyclic, so it is normal (cyclic subgroup is abelian and abelian subgroup is normal).

c) If k is even, then $x^k = e$ and $x^iy^jx^ky^l = x^iy^jey^l = x^iy^{j+l}$.

If k is odd, then $k-1$ is even and $x^k = x^{k-1} \cdot x = x$.
$xy\ = (y^{-1})x$, so $yxy = x$ or $yx = x(y^{-1})$.
Hence $y^2x = y \cdot yx = y \cdot xy^{-1} = yx \cdot y^{-1} = xy^{-1} \cdot y^{-1} = xy^{-2}$;
$y^3x = y \cdot y^2x = y \cdot x(y^{-1})^2 = yx \cdot (y^{-1})^2 = x(y^{-1}) \cdot (y^{-1})^2 = xy^{-3}$;
…
$y^ix = y \cdot y^{i-1}x = y \cdot x(y^{-1})^{i-1} = yx \cdot (y^{-1})^{i-1} = x(y^{-1}) \cdot (y^{-1})^{i-1} = xy^{-i}$;
$x^i \cdot y^j \cdot x \cdot y^l = x^i \cdot y^jx \cdot y^l = x^i \cdot xy^{-i} \cdot y^l = x^i \cdot x^k \cdot y^{l-i} = x^{k+i}y^{l-i}$.

Then $x^iy^jx^ky^l = \begin{cases} x^iy^{j+l} \text{ if } k \text{ is even;} \\ x^{k+i}y^{l-i} \text{ if } k \text{ is odd.} \end{cases}$
There are twelve elements in $D_{12} = \{e, x, y, y^2, y^3, y^4, y^5, xy, xy^2, xy^3, xy^4, xy^5\}$.
The order of x is 2 and the order of y is 6.
y^2 has order 3, as$y$ has order 6.
y^3 has order 2, as$y$ has order 6.
y^4 has order 3, as$y$ has order 6.
y^5 has order 6, as$y$ has order 6.
$xy = y^{-1}x$, so $xy \cdot xy = xy \cdot y^{-1}x = xyy^{-1}x = x^2 = e$. Then xy has order 2.
$xy^2 = xy \cdot y = y^{-1}x \cdot y = y^{-1} \cdot xy = y^{-2}\,x$, so $xy^2 \cdot xy^2 = xy^2 \cdot y^{-2}x = xy^2\,y^{-2}x = x^2 = e$.
Then xy^2 has order 2.
The similar steps can be made to prove that xy^3, xy^4, xy^5 has order 2.
Answer: The elements $x, y^3, xy, xy^2, xy^3, xy^4, xy^5$ has order 2.

d) From a) $G_3 = \{x, y^3, xy^3, e\}$ is Sylow 2-subgroup.
$G_4 = \{y^3, xy^2, xy^5, e\}$ is also Sylow 2-subgroup. We need as to show that G_4 is closed under products and inverses. The elements y^3, xy^2, xy^5 has order 2, so G_4 is closed under products.
$xy^2 \cdot y^3 = xy^5 \in G_4$;
$xy^5 \cdot y^3 = xy^8 = xy^2 \in G_4$;
$y^3 \cdot xy^2 = y^3 \cdot y^{-2}x = yx = xy^{-1} = xy^5 \in G_4$;
$y^3 \cdot xy^5 = y^3 \cdot y^{-5}x = y^{-2}x = xy^2 \in G_4$;
$xy^5 \cdot xy^2 = y^{-5}x \cdot xy^2 = y^{-5}x^2y^2 = y^{-3} = y^3 \in G_4$;
$xy^2 \cdot xy^5 = y^{-2}x \cdot xy^5 = y^{-2}x^2y^5 = y^3 \in G_4$;
Hence G_4 is closed under products.
Answer: $\{x, y^3, xy^3, e\}$, $\{y^3, xy^2, xy^5, e\}$ are Sylow 2-subgroups