Answer on Question #45358 – Math - Matrix | Tensor Analysis

Problem.

1. Find all values of a, b, c, and d for which A is skew-symmetric.

A = [[0, 2a-3b+c, 3a-5b+5c],[-2, 0, 5a-8b+6c],[-3, -5, d]]

2. Let R be the 5x5 matrix:

[[-8, 33, 38, 173, -30],[0, 0, -1, -4, 0],[0, 0, -5, -25, 1],[0, 0, 1, 5, 0],[4, -16, -19, -86, 15]]

(a) Using technology, and the characteristic polynomial of R and hence and the eigenvalues.

(b) For each of the eigenvalues, determine (by hand) how many linearly independent eigenvectors can be found.

Solution.

(a) The matrix $A = \begin{bmatrix} 0 & 2a-3b+c & 3a-5b+5c \\ -2 & 0 & 5a-8b+6c \\ -3 & -5 & d \end{bmatrix}$ is skew-symmetric if and only if $-A^T = A$ or

$$\begin{bmatrix} 0 & 2 & 3 \\ -(2a-3b+c) & 0 & 5 \\ -(3a-5b+5c) & -(5a-8b+6c) & -d \end{bmatrix} = \begin{bmatrix} 0 & 2a-3b+c & 3a-5b+5c \\ -2 & 0 & 5a-8b+6c \\ -3 & -5 & d \end{bmatrix}.$$

Then $d = 0$, $2a-3b+c = 2$, $3a-5b+5c = 3$, $5a-8b+6c = -5$

We need to solve system

$$\begin{cases} 2a-3b+c = 2; \\ 3a-5b+5c = 3; \\ 5a-8b+6c = 5. \end{cases}$$

Hence

$$\begin{cases} 2a-3b+c = 2; \\ 3a-5b+5c-5\cdot(2a-3b+c) = 3-10; \\ 5a-8b+6c-6\cdot(2a-3b+c) = 5-12. \end{cases}$$

$$\begin{cases} 2a-3b+c = 2; \\ -7a+10b = -7; \\ -7a+10b = -7. \end{cases}$$

Therefore three equation are linearly dependent and the solution is $a = t$, $b = 0.7t - 0.7$, $c = 2 - 2t + 2.1t - 2.1 = 0.1t - 0.1$, where $t \in \mathbb{R}$.

Answer: $a = t$, $b = 0.7t - 0.7$, $c = 0.1t - 0.1$, $d = 0$, $t \in \mathbb{R}$.

2.

(a) From MATLAB

```
>> R = [-8, 33, 38, 173, -30; 0, 0, -1, -4, 0; 0, 0, -5, -25, 1; 0, 0,
1, 5, 0; 4, -16, -19, -86, 15];
P = poly(R)

P =

    1.0000   -7.0000   19.0000  -25.0000   16.0000   -4.0000

>> R = roots(P)

R =
```

```
2.0000
2.0000
1.0001
0.9999 + 0.0001i
0.9999 - 0.0001i
```

Hence the characteristic polynomial is $P(\lambda) = \lambda^5 - 7\lambda^4 + 19\lambda^3 - 25\lambda^2 + 16\lambda - 4$ and the roots are $\lambda = 2$ and $\lambda = 1$. The root $\lambda = 1$ has algebraic multiplicity 3 and the root $\lambda = 2$ has algebraic multiplicity 2.

(b) For $\lambda = 2$ there are

$$5 - \text{rank}(R - 2I)$$

linearly independent eigenvectors.

$$5 - \text{rank}(R - 2I) = 5 - \text{rank}\begin{bmatrix} -10 & 33 & 38 & 173 & -30 \\ 0 & -2 & -1 & -4 & 0 \\ 0 & 0 & -7 & -25 & 1 \\ 0 & 0 & 1 & 3 & 0 \\ 4 & -16 & -19 & -86 & 13 \end{bmatrix}$$

$$\begin{bmatrix} -10 & 33 & 38 & 173 & -30 \\ 0 & -2 & -1 & -4 & 0 \\ 0 & 0 & -7 & -25 & 1 \\ 0 & 0 & 1 & 3 & 0 \\ 4 & -16 & -19 & -86 & 13 \end{bmatrix} \sim \begin{bmatrix} -10 & 33 & 38 & 173 & -30 \\ 0 & -2 & -1 & -4 & 0 \\ 0 & 0 & -7 & -25 & 1 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & -2.8 & -3.8 & -16.8 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} -10 & 33 & 38 & 173 & -30 \\ 0 & -2 & -1 & -4 & 0 \\ 0 & 0 & -7 & -25 & 1 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & -2.4 & -11.2 & 11 \end{bmatrix} \sim \begin{bmatrix} -10 & 33 & 38 & 173 & -30 \\ 0 & -2 & -1 & -4 & 0 \\ 0 & 0 & -7 & -25 & 1 \\ 0 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & -4 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} -10 & 33 & 38 & 173 & -30 \\ 0 & -2 & -1 & -4 & 0 \\ 0 & 0 & -7 & -25 & 1 \\ 0 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence $\text{rank}\begin{bmatrix} -10 & 33 & 38 & 173 & -30 \\ 0 & -2 & -1 & -4 & 0 \\ 0 & 0 & -7 & -25 & 1 \\ 0 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = 4$ and there are 1 linearly independent eigenvector.

For $\lambda = 1$ there are

$$5 - \text{rank}(R - I)$$

linearly independent eigenvectors.

$$5 - \text{rank}(R - I) = 5 - \text{rank}\begin{bmatrix} -9 & 33 & 38 & 173 & -30 \\ 0 & -1 & -1 & -4 & 0 \\ 0 & 0 & -6 & -25 & 1 \\ 0 & 0 & 1 & 4 & 0 \\ 4 & -16 & -19 & -86 & 14 \end{bmatrix}$$

$$\begin{bmatrix} -9 & 33 & 38 & 173 & -30 \\ 0 & -1 & -1 & -4 & 0 \\ 0 & 0 & -6 & -25 & 1 \\ 0 & 0 & 1 & 4 & 0 \\ 4 & -16 & -19 & -86 & 14 \end{bmatrix} \sim \begin{bmatrix} 4 & -16 & -19 & -86 & 14 \\ 0 & -1 & -1 & -4 & 0 \\ 0 & 0 & -6 & -25 & 1 \\ 0 & 0 & 1 & 4 & 0 \\ -9 & 33 & 38 & 173 & -30 \end{bmatrix}$$

$$\sim \begin{bmatrix} 4 & -16 & -19 & -86 & 14 \\ 0 & -1 & -1 & -4 & 0 \\ 0 & 0 & -6 & -25 & 1 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & -3 & -4.75 & -20.5 & 1.5 \end{bmatrix} \sim \begin{bmatrix} 4 & -16 & -19 & -86 & 14 \\ 0 & -1 & -1 & -4 & 0 \\ 0 & 0 & -6 & -25 & 1 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & -1.75 & -8.5 & 1.5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 4 & -16 & -19 & -86 & 14 \\ 0 & -1 & -1 & -4 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1.5 & 1.5 \end{bmatrix} \sim \begin{bmatrix} 4 & -16 & -19 & -86 & 14 \\ 0 & -1 & -1 & -4 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence rank $\begin{bmatrix} 4 & -16 & -19 & -86 & 14 \\ 0 & -1 & -1 & -4 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = 4$ and there are 1 linearly independent eigenvector.