Answer on Question #45230 - Math - Abstract Algebra

Question.

1. Use the Fundamental Theorem of Homomorphism to prove that $\mathbb{Z}/12\mathbb{Z} \cong \langle g \rangle$ iff g is an element of order 12 in a group G.
2. Obtain two distinct elements of $\mathbb{Z}/7\mathbb{Z}$, and two distinct subgroups of $\mathbb{Z}/7\mathbb{Z}$.

Solution.

1. Define the map $\phi : \mathbb{Z} \to \langle g \rangle$ by $\phi(n) = g^n$. It is a homomorphism, because

$$\varphi(m+n) = g^{m+n} = g^m g^n = \varphi(m)\varphi(n).$$

Moreover, it is an epimorphism, since every element of $\langle g \rangle$ has the form g^n for some $n \in \mathbb{Z}$. By the Fundamental Theorem of Homomorphism we have $\mathbb{Z}/\ker\phi \cong \langle g \rangle$.

Let $|g| = 12$. Then $g^{12} = 1$ and $g^i \neq 1$ for $1 \leq i \leq 11$, which means that $\phi(12) = 1$ and $\phi(i) \neq 1$ for $1 \leq i \leq 11$. Hence $\phi(n) = \phi(n \bmod 12) = 1$ if and only if $n \bmod 12 = 0$, that is n is divisible by 12. This shows that $\ker\phi = 12\mathbb{Z}$. Thus, $\mathbb{Z}/12\mathbb{Z} \cong \langle g \rangle$.

Conversely, if $\mathbb{Z}/12\mathbb{Z} \cong \langle g \rangle$, then $12 = |\mathbb{Z}/12\mathbb{Z}| = |\langle g \rangle| = |g|$.

2. Two distinct elements of $\mathbb{Z}/7\mathbb{Z}$ are $[0]_7$ and $[1]_7$, where $[n]_7$ means the class of n modulo 7. They are really different, because $1 - 0 = 1$ is not divisible by 7. Two distinct subgroups of $\mathbb{Z}/7\mathbb{Z}$ are $\mathbb{Z}/7\mathbb{Z}$ and the identity subgroup $\{[0]_7\}$.