Answer on Question #45095 – Math- Analytic Geometry

Problem.

Show that the angle between the two lines in which the plane x - y + 2z = 0 intersects the cone x^2 + y^2 - 4z^2 + 6yz = 0 is tan inverse of (under-root 6 / 7) .

Solution.

The lines of intersection have equation

$$\begin{cases} x^2 + y^2 - 4z^2 + 6yz = 0; \\ x - y + 2z = 0; \\ z = t, \end{cases}$$

where $t \in \mathbb{R}$.

The system

$$\begin{cases} x^2 + y^2 - 4z^2 + 6yz = 0; \\ x - y + 2z = 0; \\ z = t. \end{cases}$$

is equivalent to

$$\begin{cases} x^2 + y^2 - 4t^2 + 6yt = 0; \\ x = y - 2t; \\ z = t; \end{cases}$$

or

$$\begin{cases} y^2 - 4yt + 4t^2 + y^2 - 4t^2 + 6yt = 0; \\ x = y - 2t; \\ z = t; \end{cases}$$

or

$$\begin{cases} y(y + t) = 0; \\ x = y - 2t; \\ z = t. \end{cases}$$

We obtain two lines

$$\begin{cases} x = -2t; \\ y = 0; \\ z = t; \end{cases} \quad \text{and} \quad \begin{cases} x = -3t; \\ y = -t; \\ z = t, \end{cases}$$

where t is real parameter.

The direction vector of the first line is $v_1 = (-2,0,1)$ and the direction vector of the second line is $v_2 = (-3,-1,1)$. The angle between the two lines is equal to the angle between the direction vectors of this two lines.

$$\cos(\widehat{v_1, v_2}) = \frac{(v_1, v_2)}{|v_1| \cdot |v_2|} = \frac{6 + 1}{\sqrt{5} \cdot \sqrt{11}} = \sqrt{\frac{49}{55}}.$$

$$\tan^2(\widehat{v_1, v_2}) = \frac{1}{\cos^2(\widehat{v_1, v_2})} - 1 = \frac{55}{49} - 1 = \frac{6}{49}.$$

The $(\widehat{v_1, v_2}) = \arctan\frac{\sqrt{6}}{7}$, as $\cos(\widehat{v_1, v_2})$ is positive.

Therefore, the angle between two lines is equal to $\arctan\frac{\sqrt{6}}{7}$.