Answer on Question #45093 – Math - Analytic Geometry

Problem.

Under what conditions on (alpha) , the spheres x^2 + y^2 + z^2 + (alpha)x - y = 0 and x^2 + Y^2 + z^2 + x +2z +1 = 0 intersect each other at an angle of 45^0.

Solution.

The first sphere has equation $x^2 + y^2 + z^2 + \alpha x - y = 0$
or

$$\left(x + \frac{\alpha}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 + z^2 = \frac{\alpha^2}{4} + \frac{1}{4}.$$

Hence the first sphere has center $\left(-\frac{\alpha}{2}, \frac{1}{2}, 0\right)$ and radius $\sqrt{\frac{\alpha^2}{4} + \frac{1}{4}}$.
The second sphere has equation $x^2 + y^2 + z^2 + x + 2z + 1 = 0$
or

$$\left(x + \frac{1}{2}\right)^2 + y^2 + (z+1)^2 = \frac{1}{4}.$$

Hence the first sphere has center $\left(-\frac{1}{2}, 0, -1\right)$ and radius $\frac{1}{2}$.
Suppose that (x_0, y_0, z_0) is point from intersection of spheres. Therefore

$$x_0^2 + y_0^2 + z_0^2 + \alpha x_0 - y_0 = 0$$

and

$$x_0^2 + y_0^2 + z_0^2 + x_0 + 2z_0 + 1 = 0.$$

The angle between spheres is equal to the angle to tangent planes at point (x_0, y_0, z_0). The angle between planes is equal to the angle between normal vectors of these planes. The normal vectors of tangent planes at point (x_0, y_0, z_0) are $\left(x_0 + \frac{\alpha}{2}, y_0 - \frac{1}{2}, z_0\right)$ and $\left(x_0 + \frac{1}{2}, y_0, z_0 + 1\right)$ (these are the vectors from centers of the spheres to point) (x_0, y_0, z_0). Therefore the angle between spheres is equal to the angle between vectors $\left(x_0 + \frac{\alpha}{2}, y_0 - \frac{1}{2}, z_0\right)$ and $\left(x_0 + \frac{1}{2}, y_0, z_0 + 1\right)$.
Hence

$$\frac{x_0^2 + \frac{\alpha x_0}{2} + \frac{x_0}{2} + \frac{\alpha}{4} + y_0^2 - \frac{y_0}{2} + z_0^2 + z_0}{\sqrt{\left(x_0 + \frac{\alpha}{2}\right)^2 + \left(y_0 - \frac{1}{2}\right)^2 + z_0^2} \cdot \sqrt{\left(x_0 + \frac{1}{2}\right)^2 + y_0^2 + (z_0+1)^2}} = \cos 45^\circ = \frac{\sqrt{2}}{2}.$$

or

$$x_0^2 + y_0^2 + z_0^2 + \alpha x_0 - y_0 + x_0^2 + y_0^2 + z_0^2 + x_0 + 2z_0 = \frac{\sqrt{2}}{2} \cdot \sqrt{\frac{\alpha^2}{4} + \frac{1}{4}} - \frac{\alpha}{4}.$$

Then

$$-1 = \frac{\sqrt{2}}{2} \cdot \sqrt{\frac{\alpha^2}{4} + \frac{1}{4}} - \frac{\alpha}{4}$$

or

$$\alpha - 4 = \sqrt{2(\alpha^2 + 1)},$$

Therefore $\alpha > 4$ and

$$\alpha^2 - 8\alpha + 16 = 2\alpha^2 + 2,$$
$$\alpha^2 + 8\alpha - 14 = 0,$$
$$\alpha = \frac{-8 \pm \sqrt{64 + 4 \cdot 8 \cdot 14}}{2} = -4 \pm \sqrt{16 + 8 \cdot 14} = -4 \pm 8\sqrt{2}.$$

Hence $\alpha = 8\sqrt{2} - 4$, as $\alpha > 4$.
Answer. $\alpha = 8\sqrt{2} - 4$.