Answer on Question #45084 – Math - Analytic Geometry

Problem.

Find the equation of the plane which passes through the line of intersection of the planes 2x+y-2z=6 and 2x+3y+6z=5 and makes equal angles with these planes.

Solution.

Let α is the plane equation of which we should find and (a, b, c) is normal vector of plane α. The plane α makes equal angles with planes $2x + y - 2z = 6$ and $2x + 3y + 6z = 5$ if and only if normal vectors of α makes equal angles with normal vectors of planes $2x + y - 2z = 6$ and $2x + 3y + 6z = 5$ or when cosines of these angles are equal. The normal vectors of planes $2x + y - 2z = 6$ and $2x + 3y + 6z = 5$ are $(2,1,-2)$ and $(2,3,6)$ respectivly. Let φ_1 is angle between (a, b, c) and $(2,1,-2)$ and φ_2 is angle between (a, b, c) and $(2,3,6)$. From inner product formula

$$\cos\varphi_1 = \frac{2a + b - 2c}{\sqrt{a^2 + b^2 + c^2} \cdot \sqrt{2^2 + 1^2 + (-2)^2}},$$

$$\cos\varphi_2 = \frac{2a + 3b + 6c}{\sqrt{a^2 + b^2 + c^2} \cdot \sqrt{2^2 + 3^2 + 6^2}},$$

So $\frac{2a+b-2c}{\sqrt{a^2+b^2+c^2}\cdot\sqrt{2^2+1^2+(-2)^2}} = \frac{2a+3b+6c}{\sqrt{a^2+b^2+c^2}\cdot\sqrt{2^2+3^2+6^2}}$ $(\cos\varphi_1 = \cos\varphi_2)$. Hence

$$14a + 7b - 14c = 6a + 9b + 18c$$

or

$$8a - 2b - 32c = 0 \text{ or } b = 4a - 16c.$$

The line of intersection of the planes $2x + y - 2z = 6$ and $2x + 3y + 6z = 5$ has equation

$$\begin{cases} 2x + y - 2z = 6; \\ 2x + 3y + 6z = 5; \\ z = t, \end{cases}$$

where $t \in \mathbb{R}$.

Then $y = -\frac{8t+1}{2}$ and $x = \frac{12t+13}{4}$.

Then the equation of plane α is

$$a\left(x - \frac{12t + 13}{4}\right) + b\left(y + \frac{8t + 1}{2}\right) + c(z - t) = 0$$

or

$$ax + by + cz - \frac{13a}{4} + \frac{b}{2} + (-3a + 4b - c)t = 0$$

is for all $t \in \mathbb{R}$.

Hence for all t

$$(-3a + 4b - c)t = 0.$$

Therefore

$$3a - 4b + c = 0,$$

Hence $3a - 4 \cdot (4a - 16c) + c = 0$ (as $b = 4a - 16c$), $-13a + 65c = 0$, $a = \frac{65}{13}c$. Then $a = \frac{65}{13}c$ and $b = 4c$. The equation of α plane is

$$\frac{65}{13}x + 4y + z - \frac{65}{4} + 2 = 0;$$

$$\frac{65}{13}x + 4y + z - \frac{57}{4} = 0.$$

Answer: $\frac{65}{13}x + 4y + z - \frac{57}{4} = 0$