Answer on Question #45082 – Math - Analytic Geometry

Problem.

For what value(s) of α, the conicoid x^2+y^2+αz^2+2yz+xy+x+2y+z+3=0 has a unique centre? Give reason for your answer.

Solution.

Denoting the given expression by $F(x, y, z)$ we get from

$$\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial z} = 0$$

or

$$2x + y + 1 = 0,$$
$$2y + x + 2 = 0,$$
$$2\alpha z + 2y + 1 = 0.$$

Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 2 & 2\alpha \end{bmatrix}$ and $A' = \begin{bmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 0 & 2 \\ 0 & 2 & 2\alpha & 1 \end{bmatrix}$.

The conicoid $x^2 + y^2 + \alpha z^2 + 2yz + xy + x + 2y + z + 3 = 0$ has a unique centre if $\operatorname{rank} A = \operatorname{rank} A' = 3$.

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 2 & 2\alpha \end{bmatrix} \sim \begin{bmatrix} 0 & -3 & 0 \\ 1 & 2 & 0 \\ 0 & 2 & 2\alpha \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & -3 & 0 \\ 0 & 2 & 2\alpha \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2\alpha \end{bmatrix}$$

Hence $\operatorname{rank} A = 3$ if $\alpha \neq 0$.

$$A' = \begin{bmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 0 & 2 \\ 0 & 2 & 2\alpha & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & -3 & 0 & -3 \\ 1 & 2 & 0 & 2 \\ 0 & 2 & 2\alpha & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & -3 & 0 & -3 \\ 0 & 2 & 2\alpha & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & -3 & 0 & -3 \\ 0 & 0 & 2\alpha & -1 \end{bmatrix}$$

Hence $\operatorname{rank} A' = 3$.

Therefore conicoid has a unique centre if $\alpha \neq 0$.

Answer: $\alpha \neq 0$.