# Answer on Question #45072 – Math - Analytic Geometry

**Problem.**

Check whether the following statements are true or false. Justify your answer with a short explanation or a counter example.
(i) Any line through the origin cuts the sphere x^2+y^2+z^2=4 at exactly two points.
(ii) The plane making intercept at the z-axis and parallel to the xy-plane intersects the cone x^2+y^2 = z^2(tan theta)^2 in a circle.
(iii) There exists no line with 1/under-root3 ,1/under-root2 ,1/under-root6 as direction cosines.
(iv) The tangent planes at the extremities of any axis of an ellipsoid are perpendicular.
(v) A section of an elliptic paraboloid by a plane is always an ellipse.
(vi) The curve xy^2+yx^2=0 is symmetric about the origin.
(vii) There exists a unique line which is perpendicular to the lines x=y=z/2 and x=y= -z.
(viii) The plane 3x+4y+2z=1 touches the conicoid 3x^2+2y^2=z^2=1.
(ix) The xy- plane intersects the sphere x^2+y^2+z^2+2x-z=2 in a great circle.
(x) Non degenerate conics are non-central.

**Solution.**

(i) The statement is true.
The lines through the origin has equation $x = \alpha t$, $y = \beta t$, $z = \gamma t$ (where $\alpha, \beta, \gamma \in \mathbb{R}$). This line intersect sphere at $(\alpha t_0, \beta t_0, \gamma t_0)$ and $(-\alpha t_0, -\beta t_0, -\gamma t_0)$ (where $(\alpha^2 + \beta^2 + \gamma^2)t_0^2 = 4$).
(ii) The statement is false.
The equation of xy-plane is $z = 0$. Hence if $(x_0, y_0, z_0)$ is the point from the intersection, then $z_0 = 0$ and $x_0^2 + y_0^2 = z_0^2 \tan^2\theta = 0$ or $x_0 = 0$ and $y_0 = 0$. Therefore xy-plane intersects the cone $x^2 + y^2 = z^2(\tan\theta)^2$ in a point.
(iii) The statement is false.
The line $\frac{x}{1/\sqrt{3}} = \frac{y}{1/\sqrt{2}} = \frac{z}{1/\sqrt{6}}$ has direction vector $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}\right)$. $\left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{6}}\right)^2 = 1$, so direction the direction cosines are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}$.
(iv) The statement is false.
The tangent planes at the extremities of a x-axis of an ellipsoid $x^2 + y^2 + z^2 = 1$ are $x = 1$ and $x = -1$. This planes are parallel.
(v) The statement is false.
$z = x^2 + y^2$ is elliptic paraboloid. The intersection of $z = x^2 + y^2$ with plane $y = 0$ is $z = x^2$.
(vi) The statement is true.
Let $l: xy^2 + yx^2 = 0$. If $(x, y) \in l$, then $(-x, -y) \in l$ (as $xy^2 + yx^2 = 0 = -(xy^2 + yx^2) = (-x)(-y)^2 + (-y)(-x)^2$). Threrefore the curve is symmetric about the origin.
(vii) The statement is true.
Suppose that the direction vector of line $l$ perpendicular to the lines $x = y = \frac{z}{2}$ and $x = y = -z$ is $(a, b, c)$. Then $1 \cdot a + 1 \cdot b + 2 \cdot c = 0$ and $1 \cdot a + 1 \cdot b - c = 0$. Hence $c = 0$ and $a + b = 0$. Therefore the direction vector of line $l$ is $(1, -1, 0)$. If $(x_0, y_0, z_0)$ is point of intersection of lines $l$ and $x = y = -z$, then $(x_0, y_0, z_0) = (x_0, x_0, -x_0)$. Hence the line $l$ has equation $x = t + x_0$, $y = -t + x_0$, $z = -x_0$. There should exist point of intersection of $l$ and $x = y = \frac{z}{2}$. Hence there should exist $t_0$ such that $t_0 + x_0 = -t_0 + x_0 = -\frac{x_0}{2}$, so $t_0 = x_0 = 0$. Hence the equation of $l$ is $x = t$, $y = -t$, $z = 0$, so there exist a unique line.
(viii) The statement is false.
The coincoid $3x^2 + 2y^2 = z^2 = 1$ is union two curves (ellipses).
(ix) The statement is true.

The intersection is curve $z = 0$, $x^2 + y^2 + 2x = 2$ or $(x + 1)^2 + y^2 = (\sqrt{3})^2$ it is circle.
(x) The statement is false.
The conic $x^2 + y^2 + z^2 = 1$ is non degenerate and has center $(0,0,0)$.