Answer on Question #44918 – Math - Linear Algebra

Problem.

State if the following statements are true and which are false? Justify your answer with a short proof or a counterexample.

If {v1, v2, ... , vn} is a basis for vector space V, then {v1+v2+ +vn, v2,.... ,vn} is also a basis for V

Solution.

The statement is true.

We need to show that $\{v_1 + v_2 + \cdots + v_n, v_2, \dots, v_n\}$ is a set of linearly independent vectors and each element of V can be represented as linear combination of vectors from set.

Suppose that vectors $v_1 + v_2 + \cdots + v_n, v_2, \dots, v_n$ are linearly dependent. Then there exist $\lambda_1, \lambda_2, \dots, \lambda_n \in P$ (where P is a field) such that

$$\lambda_1(v_1 + v_2 + \cdots + v_n) + \lambda_2 v_2 + \cdots + \lambda_n v_n = 0$$

or

$$\lambda_1 v_1 + (\lambda_1 + \lambda_2)v_2 + \cdots + (\lambda_1 + \lambda_n)v_n = 0,$$

but $\lambda_1 v_1 + (\lambda_1 + \lambda_2)v_2 + \cdots + (\lambda_1 + \lambda_n)v_n \neq 0$, as vectors $v_1, v_2, \dots, v_n$ are linearly independent.

Hence $v_1 + v_2 + \cdots + v_n, v_2, \dots, v_n$ are linearly independent.

If $u \in V$, then there exist $\lambda_1, \lambda_2, \dots, \lambda_n \in P$ ($\{v_1, v_2, \dots, v_n\}$ is a basis), such that

$$u = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n$$

or

$$u = \lambda_1(v_1 + v_2 + \cdots v_n) + (\lambda_2 - \lambda_1)v_2 + \cdots + (\lambda_n - \lambda_1)v_n.$$

Hence each element of V can be represented as linear combination of vectors from set $\{v_1 + v_2 + \cdots + v_n, v_2, \dots, v_n\}$.