Answer on Question #44737 – Math – Abstract Algebra:

Let $\sigma = (a_1\ a_2 \dots a_k) \in S_n$ be a cycle. Let $\tau \in S_n$. Check that $\tau\sigma\tau^{-1} = (b_1\ b_2\ \cdots\ b_k)$, where $\tau\,(a_i) = b_i$.

Solution.

Denote $\alpha = \tau\sigma\tau^{-1}$. We need to prove that:

$$\forall i = 1, \dots, k-1: \alpha(b_i) = b_{i+1};$$

$$\alpha(b_k) = b_1;$$

$$\forall x \in \{1, \dots, n\} \backslash \{b_1, \dots, b_k\}: \alpha(x) = x;$$

So:

$$1 \le i \le k-1 \Rightarrow \alpha(b_i) = (\tau\sigma\tau^{-1})(b_i) = \tau\left(\sigma\left(\tau^{-1}(b_i)\right)\right) = \tau\left(\sigma(a_i)\right) = \tau(a_{i+1}) = b_{i+1};$$

$$\alpha(b_k) = (\tau\sigma\tau^{-1})(b_k) = \tau\left(\sigma\left(\tau^{-1}(b_k)\right)\right) = \tau\left(\sigma(a_k)\right) = \tau(a_1) = b_1;$$

$$x \notin \{b_1, \dots, b_k\}, x = \tau(y) \Rightarrow y \notin \{a_1, \dots a_k\} \Rightarrow \sigma(y) = y \Rightarrow$$

$$\Rightarrow \alpha(x) = (\tau\sigma\tau^{-1})(x) = \tau\left(\sigma\left(\tau^{-1}(x)\right)\right) = \tau\left(\sigma(y)\right) = \tau(y) = x.$$