Answer on Question #44732 – Math – Abstract Algebra

Show that the map $f : \mathbb{Z} + i\mathbb{Z} \to \mathbb{Z}_2$, defined by $f(a+ib) = (a-b)\,(mod\ 2)$, is an onto ring homomorphism. Describe $Ker(f)$. Is it a maximal ideal? Justify your answer.

Solution.

We need to prove that $\forall x, y \in \mathbb{Z} + i\mathbb{Z} : f(x+y) = f(x) + f(y), f(x \cdot y) = f(x)f(y)$.

$$x, y \in \mathbb{Z} + i\mathbb{Z} \Rightarrow x = a_1 + ib_1, y = a_2 + ib_2, a_1, a_2, b_1, b_2 \in \mathbb{Z};$$

$$f(x+y) = f(a_1 + ib_1 + a_2 + ib_2) = f\big(a_1 + a_2 + i(b_1 + b_2)\big) =$$

$$= (a_1 + a_2 - b_1 - b_2)(mod\ 2) = (a_1 - b_1 + a_2 - b_2)(mod\ 2) = f(x) + f(y);$$

$$f(x \cdot y) = f\big((a_1 + ib_1)(a_2 + ib_2)\big) = f\big(a_1a_2 - b_1b_2 + i(a_1b_2 + a_2b_1)\big) =$$

$$= (a_1a_2 - b_1b_2 - a_1b_2 - a_2b_1)(mod\ 2) = (a_1a_2 - b_1b_2 - a_1b_2 - a_2b_1 + 2b_1b_2)(mod\ 2) =$$

$$= (a_1a_2 + b_1b_2 - a_1b_2 - a_2b_1)(mod\ 2) = \big(a_1(a_2 - b_2) - b_1(a_2 - b_2)\big)(mod\ 2) =$$

$$= \big((a_1 - b_1)(a_2 - b_2)\big)(mod\ 2) = f(x)f(y);$$

So, f is a homomorphism of rings.

$$f(0) = (0 - 0)(mod\ 2) = 0;$$

$$f(1) = (1 - 0)(mod\ 2) = 1;$$

All elements of $\mathbb{Z}_2$ are images of f, so f is a homomorphism onto.

Find $Ker(f)$:

$$f(a + ib) = 0 \Rightarrow a - b \equiv 0(mod\ 2) \Rightarrow a \equiv b(mod\ 2);$$

Hence:

$$Ker(f) = \{a + ib \mid a \equiv b(mod\ 2)\}.$$

Prove that $Ker(f)$ is a maximal ideal.

Assume the contrary, i.e. there exists ideal $J \neq Ker(f)$ such that $1 \notin J$ and $Ker(f) \subset J$.

$$J \neq Ker(f) \Rightarrow \exists\ a + ib \in J : a + ib \notin Ker(f);$$

$$a + ib \notin Ker(f) \Rightarrow a - b \equiv 1(mod\ 2) \Rightarrow \left[\begin{array}{l} a = 2n, b = 2m + 1 \\ a = 2n + 1, b = 2m \end{array}\right.;$$

Hence:

$$\left[\begin{array}{c} 2n + i(2m+1) \in J \\ 2n + 1 + i2m \in J \end{array}\right. \Rightarrow \left[\begin{array}{c} 2n + i(2m+1) - \big(2n - 1 + i(2m+1)\big) \in J \\ 2n + 1 + i2m - (2n + i2m) \in J \end{array}\right. \Rightarrow$$

$$\Rightarrow \begin{bmatrix} 1 \in J \\ 1 \in J \end{bmatrix} \Rightarrow 1 \in J - \text{contradiction.}$$

So, our assumption doesn't hold and $Ker(f)$ is a maximal ideal.