Answer to Question #44670 - Math - Functional Analysis

Let M be a sequence of the Hilbert space X. Prove that M is dense in X iff

$$(Y \perp M) \Rightarrow y = 0$$

Consider that M is dense in X. If it is, then $\overline{M} = X$. Then, $M \perp Y$ means that $(m, y) = 0 \forall m \in M \forall y \in Y$. As $\overline{M} = X$, then $\forall x \in X \exists m_k : m_k \to x$. Therefore, if we consider a sequence

$$(m_k, y), k \in \mathbb{N},$$

then, because of the fact that dot product is a continuous function of one multiplier,

$$(m_k, y) \to (x, y) \quad \forall x \in X (k \to \infty).$$

Therefore,

$$(x, y) = 0 \quad \forall x \in X \forall y \in Y$$

As $Y \subset X$, this equality can hold only in case when $y = 0$, because x changes through all possible values in X