Answer on Question #44669 – Math - Trigonometry

Prove that in a triangle with angles A, B and C; and sides of length a, b and c that:

1/[(a-b)(a-c)]*tan(A/2) + 1/[(b-c)(b-a)]*tan(B/2) + 1/[(c-b)(c-a)]*tan(C/2) =(Area of triangle)^(-1)

{NOTE: 'a' is the length of side opposite to angle A, likewise 'b' is the length of side opposite to angle B and similarly 'c' is the length of side opposite to angle C.}

Solution.

According the Law of cosines: $\cos A = \frac{b^2+c^2-a^2}{2bc}$

As we know

$\cos A = 2\cos^2\frac{A}{2} - 1 \rightarrow \cos^2\frac{A}{2} = \frac{1+\cos A}{2} = \frac{b^2+c^2-a^2+2bc}{4bc} = \frac{(b+c-a)(b+c+a)}{4bc} = \frac{s(s-a)}{bc}$, **where** $s = \frac{a+b+c}{2}$.

$$\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}},\quad \sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}},$$

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \frac{P}{s(s-a)}$$

Where $P = \sqrt{s(s-a)(s-b)(s-c)}$ **- area of the triangle (Heron's formula).**

Similarly: $\tan\frac{B}{2} = \frac{P}{s(s-b)}$, $\tan\frac{C}{2} = \frac{P}{s(s-c)}$.

So, $\frac{\tan\frac{A}{2}}{(a-b)(a-c)} + \frac{\tan\frac{B}{2}}{(b-c)(b-a)} + \frac{\tan\frac{C}{2}}{(c-b)(c-a)} =$

$$= \frac{P}{s}\left[\frac{1}{(a-b)(a-c)(s-a)} + \frac{1}{(b-a)(b-c)(s-b)} + \frac{1}{(c-a)(c-b)(s-c)}\right] =$$

$$= \frac{P}{s} * \frac{-(b-c)(s-b)(s-c)-(c-a)(s-a)(s-c)-(a-b)(s-a)(s-b)}{(a-b)(b-c)(c-a)(s-a)(s-b)(s-c)} =$$

$$= -\frac{P}{s(s-a)(s-b)(s-c)} *$$

$$* \frac{(b-c)(a-b+c)(a+b-c)+(c-a)(b+c-a)(a+b-c)+(a-b)(b+c-a)(a-b+c)}{4(a-b)(b-c)(c-a)}$$

$$= -\frac{P}{P^2} * \frac{4ac^2+4a^2b-4a^2c+4b^2c-4bc^2+4ab^2}{4(a-b)(b-c)(c-a)} = \frac{1}{P} * \frac{4(a-b)(b-c)(c-a)}{4(a-b)(b-c)(c-a)} =$$

$$= \frac{1}{P} \ .$$

Q.E.D.