Answer on Question #44643 – Math – Linear Algebra:

Let $T: R^2 \to R^2$ and $S: R^2 \to R^2$ be linear operators defined by:

$$T(x_1, x_2) = (x_1 + x_2, x_1 - x_2);$$
$$S(x_1, x_2) = (x_1, x_1 + 2x_2);$$

(a) Find T∘S and S∘T.

(b) Let $B = \{(1,0)^T, (0,1)^T\}$ be the standard basis of R^3. Verify that $[T \circ S]B = [T]B \circ [S]B$.

Solution.

(a)

$$(T \circ S)(x_1, x_2) = T\big(S(x_1, x_2)\big) =$$
$$= T(x_1, x_1 + 2x_2) = \big(x_1 + (x_1 + 2x_2), x_1 - (x_1 + 2x_2)\big) = (2x_1 + 2x_2, -2x_2);$$
$$(S \circ T)(x_1, x_2) = S\big(T(x_1, x_2)\big) =$$
$$= S(x_1 + x_2, x_1 - x_2) = \big(x_1 + x_2, x_1 + x_2 + 2(x_1 - x_2)\big) = (x_1 + x_2, 3x_1 - x_2);$$

(b)

Denote $e_1 = (0,1)^T, e_2 = (1,0)^T$.

$$Te_1 = (1 + 0, 1 - 0) = (1,1);$$
$$Te_2 = (0 + 1, 0 - 1) = (1, -1);$$
$$Se_1 = (1, 1 + 2 \cdot 0) = (1,1);$$
$$Se_2 = (0, 0 + 2 \cdot 1) = (0,2);$$
$$(T \circ S)e_1 = (2 \cdot 1 + 2 \cdot 0, -2 \cdot 0) = (2,0);$$
$$(T \circ S)e_2 = (2 \cdot 0 + 2 \cdot 1, -2 \cdot 1) = (2, -2);$$

Hence:

$$(T \circ S)(x_1, x_2) = \begin{pmatrix} 2 & 2 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix};$$
$$T(x_1, x_2) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix};$$
$$S(x_1, x_2) = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix};$$
$$[T]B \circ [S]B = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 \cdot 1 + 1 \cdot 1 & 1 \cdot 0 + 1 \cdot 2 \\ 1 \cdot 1 + (-1) \cdot 1 & 1 \cdot 0 + (-1) \cdot 2 \end{pmatrix} =$$
$$= \begin{pmatrix} 2 & 2 \\ 0 & -2 \end{pmatrix} = [T \circ S]B.$$