Answer on Question #44642, Math, Linear Algebra

Let v be the vector space of polynomial with real coefficients and of degree at most 2. If D=d/dx is the differential operator on v and B={1+2x^2, x+x^2, x^2} is an ordered basis of V, find [D]b. find the rank and nullity of D. Is D invertible? Justify your answer.

Solution

Let $B_1 = 1 + 2x^2, B_2 = x + x^2, B_3 = x^2$. Then

$$DB_1 = \frac{d}{dx}(1 + 2x^2) = 4x, DB_2 = \frac{d}{dx}(x + x^2) = 1 + 2x, DB_3 = \frac{d}{dx}(x^2) = 2x.$$

We can see that

$$DB_1 = 4B_2 - 4B_3,$$

$$DB_2 = (B_1 - 2B_3) + 2B_2 - 2B_3 = B_1 + 2B_2 - 4B_3,$$

$$DB_3 = 2B_2 - 2B_3.$$

So we get

$$D\begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} 0 & 4 & -4 \\ 1 & 2 & -4 \\ 0 & 2 & -2 \end{pmatrix}\begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}.$$

The differential operator on v with basis $B = \{1 + 2x^2, x + x^2, x^2\}$ expressed by matrix

$$D = \begin{pmatrix} 0 & 4 & -4 \\ 1 & 2 & -4 \\ 0 & 2 & -2 \end{pmatrix}.$$

Observe that rows 1 and 3 are linearly dependent, with

$$\mathrm{row}_1(D) = 2 \cdot \mathrm{row}_3(D).$$

The rank of D is number of linearly independent rows of D. So

$$R(D) = 2.$$

The nullity of D is

$$N(D) = 3 - R(D) = 1.$$

The determinant of D is zero, because rows 1 and 3 are linearly dependent. That's why differential operator D is not invertible.