

## Answer on Question #44600 – Mathematics – Analytic Geometry

### Question:

Vertices  $B$  and  $C$  of a  $\Delta ABC$  lie along the line

2 1 0

2 1 4

x y z.

Find the area of the triangle given, that  $A$  has coordinates  $\{1, 1, 2\}$  and line segment  $BC$  has length 5 units.

### Solution:

To compute the area of a triangle formed by three points  $A$ ,  $B$  and  $C$  in space (see fig.1) we use the vector product.

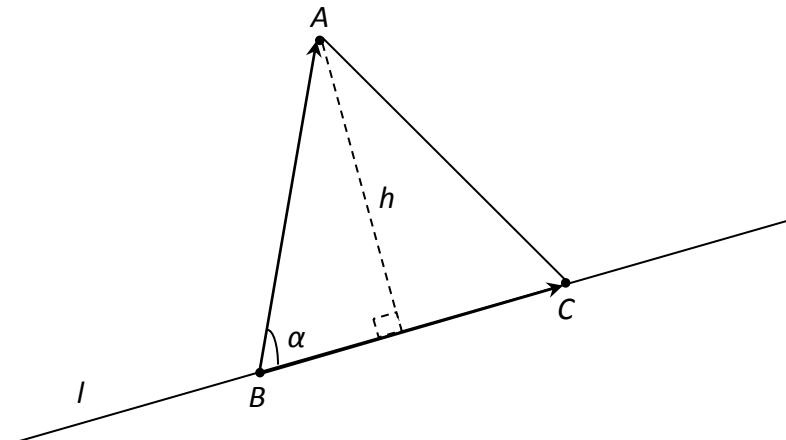


Fig. 1

By definition the area of this triangle is

$$S = \frac{1}{2} |\overrightarrow{BA} \times \overrightarrow{BC}| = \frac{1}{2} |\overrightarrow{BA}| \cdot |\overrightarrow{BC}| \sin \alpha, \quad (1)$$

where  $\alpha$  is the angle between the vectors  $\overrightarrow{BA}$  and  $\overrightarrow{BC}$ . Taking  $\overrightarrow{BC}$  to be the base of the triangle  $ABC$ , then the height of the triangle is  $h = |\overrightarrow{BA}| \sin \alpha$ . Therefore, we can rewrite (1) in the following form

$$S = \frac{1}{2} |\overrightarrow{BA} \times \overrightarrow{BC}| = \frac{1}{2} h \cdot BC, \quad (1a)$$

where  $BC = |\overrightarrow{BC}|$ .

Let's find the height  $h$  in two steps.

1) Write the canonical equation of the line  $l$  passing through the two given points  $M_1(2, 1, 0)$  and  $M_2(2, 1, 4)$ .

2) Determine the distance  $h$  from the given point  $A(1, 1, 2)$  to the line  $l$ .

The canonical equation of the line passing through the two points in space is given by relation:

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}. \quad (2)$$

Substituting into (2) the coordinates of points  $M_1$  and  $M_2$  we obtain the equation of the line  $l$

$$\frac{x-2}{0} = \frac{y-1}{0} = \frac{z}{4}. \quad (3)$$

From (2) and (3) it is easy to see that  $\overrightarrow{M_1M_2} = \{x_2 - x_1, y_2 - y_1, z_2 - z_1\} = \{0, 0, 4\}$  is the directing vector of the line  $l$  (fig.2).

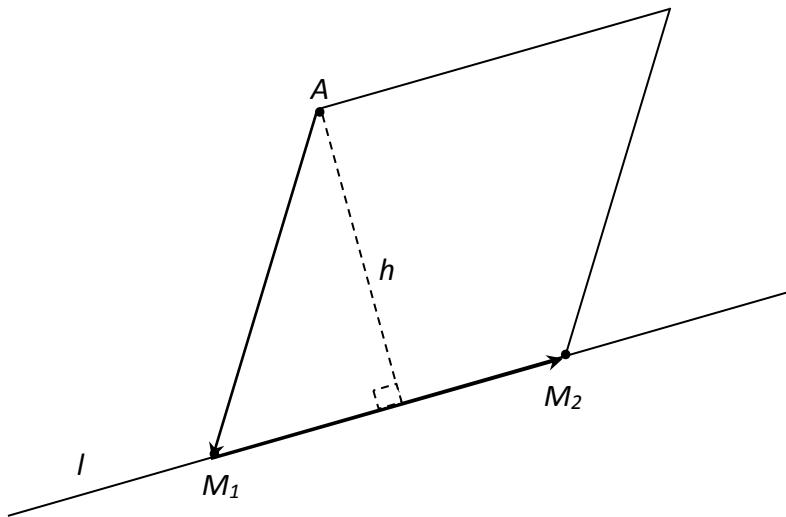


Fig. 2

Hence, the distance between point  $A$  and line  $l$  can be found using the formula that follows from definition of parallelogram area:

$$h = \frac{|\overrightarrow{AM_1} \times \overrightarrow{M_1M_2}|}{|\overrightarrow{M_1M_2}|}. \quad (4)$$

Let's calculate the numerator and denominator of (4) separately.

$$\overrightarrow{AM_1} = \{x_1 - x_A, y_1 - y_A, z_1 - z_A\} = \{1, 0, -2\},$$

$$\begin{aligned} \overrightarrow{AM_1} \times \overrightarrow{M_1M_2} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -2 \\ 0 & 0 & 4 \end{vmatrix} = (0 \cdot 4 - 0 \cdot (-2))\vec{i} - (1 \cdot 4 - 0 \cdot (-2))\vec{j} + (1 \cdot 0 - 0 \cdot 0)\vec{k} = 4\vec{j} = \\ &= \{0, -4, 0\}, \end{aligned}$$

$$|\overrightarrow{AM_1} \times \overrightarrow{M_1M_2}| = \sqrt{0^2 + (-4)^2 + 0^2} = 4,$$

$$|\overrightarrow{M_1M_2}| = \sqrt{0^2 + 0^2 + (4)^2} = 4.$$

From here we have:  $h = \frac{4}{4} = 1$  unit.

Therefore, the area of the triangle  $ABC$  is equal to

$$S = \frac{1}{2} h \cdot BC = \frac{1}{2} \cdot 1 \cdot 5 = 2.5 \text{ square units.}$$

**Answer:** the area of the triangle  $ABC$  is equal to 2.5 square units.