Answer on Question #44600 – Mathematics – Analytic Geometry

Question:

Vertices B and C of a ΔABC lie along the line

2 1 0

2 1 4

x y z.

Find the area of the triangle given, that A has coordinates {1, 1, 2} and line segment BC has length 5 units.

Solution:

To compute the area of a triangle formed by three points A, B and C in space (see fig.1) we use the vector product.

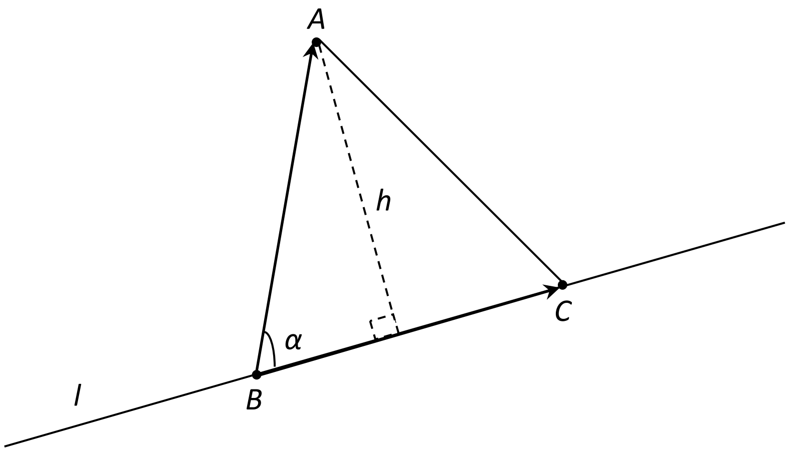

Fig. 1

By definition the area of this triangle is

$$S = \frac{1}{2}\left|\overrightarrow{BA} \times \overrightarrow{BC}\right| = \frac{1}{2}\left|\overrightarrow{BA}\right| \cdot \left|\overrightarrow{BC}\right| sin\alpha, \quad (1)$$

where α is the angle between the vectors $\overrightarrow{BA}$ и $\overrightarrow{BC}$. Taking $\overrightarrow{BC}$ to be the base of the triangle ABC, then the height of the triangle is $h = \left|\overrightarrow{BA}\right| sin\alpha$. Therefore, we can rewrite (1) in the following form

$$S = \frac{1}{2}\left|\overrightarrow{BA} \times \overrightarrow{BC}\right| = \frac{1}{2} h \cdot BC, \quad (1a)$$

where $BC = \left|\overrightarrow{BC}\right|$.

Let's find the height h in two steps.

1) Write the canonical equation of the line *l* passing through the two given points $M_1 (2, 1, 0)$ and $M_2 (2, 1, 4)$.

2) Determine the distance *h* from the given point A (1, 1, 2) to the line *l*.

The canonical equation of the line passing through the two points in space is given by relation:

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}. \tag{2}$$

Substituting into (2) the coordinates of points M_1 and M_2 we obtain the equation of the line *l*

$$\frac{x-2}{0} = \frac{y-1}{0} = \frac{z}{4}. \tag{3}$$

From (2) and (3) it is easy to see that $\overrightarrow{M_1M_2} = \{x_2 - x_1, y_2 - y_1, z_2 - z_1\} = \{0, 0,4\}$ is the directing vector of the line *l* (fig.2).

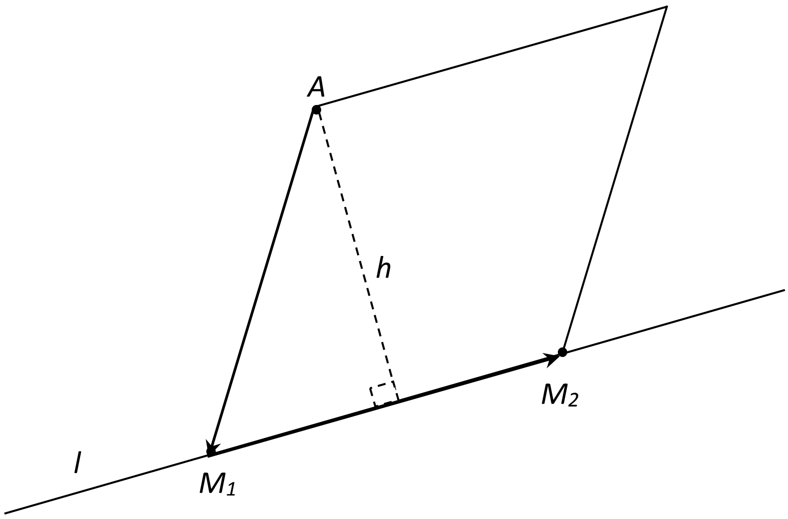

Fig. 2

Hence, the distance between point *A* and line *l* can be found using the formula that follows from definition of parallelogram area:

$$h = \frac{\left|\overrightarrow{AM_1} \times \overrightarrow{M_1M_2}\right|}{\left|\overrightarrow{M_1M_2}\right|}. \tag{4}$$

Let's calculate the numerator and denominator of (4) separately.

$$\overrightarrow{AM_1} = \{x_1 - x_A, y_1 - y_A, z_1 - z_A\} = \{1, 0, -2\},$$

$$\overrightarrow{AM_1} \times \overrightarrow{M_1M_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -2 \\ 0 & 0 & 4 \end{vmatrix} = \left(0 \cdot 4 - 0 \cdot (-2)\right)\vec{i} - \left(1 \cdot 4 - 0 \cdot (-2)\right)\vec{j} + (1 \cdot 0 - 0 \cdot 0)\vec{k} = 4\vec{j} =$$
$$= \{0, -4,0\},$$

$$\left|\overrightarrow{AM_1} \times \overrightarrow{M_1M_2}\right| = \sqrt{0^2 + (-4)^2 + 0^2}=4,$$

$$\left|\overrightarrow{M_1M_2}\right| = \sqrt{0^2 + 0^2 + (4)^2}=4.$$

From here we have: $h = \frac{4}{4} = 1$ unit.

Therefore, the area of the triangle *ABC* is equal to

$$S = \frac{1}{2}h \cdot BC = \frac{1}{2} \cdot 1 \cdot 5 = 2.5 \text{ square units.}$$

Answer: the area of the triangle *ABC* is equal to 2.5 square units.