Answer on Question #44598 – Math - Other

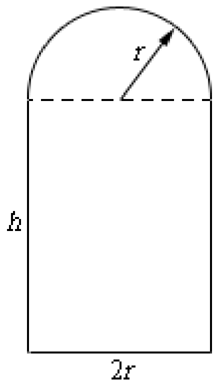
(1)A window is in the form of rectangle surmounted by a semi-circle. The total perimeter of the window is 10 m. Find the dimensions of the window to admit maximum light through the whole opening.

Solution:

We start to solve with given data. We know that a window in the shape described above to have a maximum area (and hence let in the most light) and have a perimeter of 10 m.

It also should be noted the reasonable assumption that the maximum amount of light would be admitted by the maximum area window.

Represent the sketch of the window below.



Let y or h represent the measure of the vertical dimension of the rectangular portion of the window. Let x represent the horizontal dimension of the window. The value of x is also the diameter of the semi-circular part of the window so we can write that $x = 2r$.

Then the perimeter of the window will be equal:

$$P(x, y) = 2y + x + \frac{\pi x}{2}$$

We know that the total perimeter of the window is equal to 10 meters. So, we can substitute the value of perimeter into the given formula:

$$2y + x + \frac{\pi x}{2} = 10$$

Now we can solve for y.

$$2y = 10 - x - \frac{\pi x}{2}$$

Divide both sides of the equation by 2 and simplify the equation.

$$y = 5 - \frac{x}{2} - \frac{\pi x}{4}$$

$$y = 5 - \frac{(\pi + 2)x}{4}$$

The area of the window is consisting of area semicircle and area of rectangle. So, we can write the formula for determination the area of the given window.

$$A(x, y) = xy + \frac{\pi\frac{x^2}{2}}{2}$$

Then we substitute y to create a function for area in terms of x.

$$A(x, y) = x(5 - \frac{(\pi + 2)x}{4}) + \frac{\pi\frac{x^2}{2}}{2}$$

Simplify the equation.

$$A(x, y) = 5x - \frac{(\pi + 2)x^2}{4} + \frac{\pi\frac{x^2}{2}}{2}$$

Finally simplify the value of Area.

$$A(x, y) = 5x - \frac{(4 + \pi)}{8}x^2$$

Then we can apply two methods of solution. First implies the solution by consideration the quadratic equation or algebraically solve equation.

So, in our case we have a quadratic with a negative lead coefficient, hence the graph is a parabola that opens downward. That means the vertex is a maximum. It should also be noted that the vertex of $p(x) = ax^2 + bx + c$ is at $x_v = -\frac{b}{2a}$.

Applying this rule for our problem we can find the value of x_v.

$$x_{max} = -\frac{b}{2a} = \frac{-5}{-2\left(\frac{(4 + \pi)}{8}\right)} = \frac{5}{\frac{(4 + \pi)}{4}} = \frac{20}{4 + \pi} \approx 2.801 \text{ meter.}$$

Now we can find the value of y. We have the formula for $y = 5 - \frac{(\pi+2)x}{4}$, so we can substitute the value of x.

$$y = 5 - \frac{(\pi + 2)}{4}\frac{20}{(4 + \pi)}$$

$$y_{max} = \frac{5(\pi + 2)}{(4 + \pi)} \approx 3.599 \text{ meter.}$$

Then find the radius of the semi-circle is then:

$$\frac{x_{max}}{2} = \frac{10}{4+\pi} \approx 1.401 \text{ meter.}$$

Now we can consider another method of solution. We have a second degree polynomial equation, hence continuous and twice differentiable over the real numbers. Therefore there will be a local extremum at any point where the first derivative is equal to zero and that point will be a maximum if the second derivative is negative at that point.

$$\frac{dA}{dx} = 5 - \frac{(4+\pi)}{4}\mathrm{x}$$

$$\frac{dA}{dx} = 0$$

$$5 - \frac{(4+\pi)}{4}\mathrm{x} = 0$$

Simplify by solving for x.

$$-\frac{(4+\pi)}{4}\mathrm{x} = -5$$

Multiply both sides of the equation by -4.

$$(4+\pi)\mathrm{x} = 20$$

The value of x will be equal.

$$\mathrm{x} = \frac{20}{4+\pi}$$

Then we find the second derivative.

$$\frac{d^2A}{dx^2} = -\frac{(4+\pi)}{4} < 0$$

Hence value of x is equal to $\mathrm{x} = \frac{20}{4+\pi}$ and the -coordinate of the maximum point.

The y dimension and the semi-circle radius are calculated as above.

Answer: The dimensions of the window to admit maximum light through the whole opening will be the horizontal dimension of the window $x_{max} = \frac{20}{4+\pi} \approx 2.801$ meter, radius of the semi-circle $\frac{x_{max}}{2} = \frac{10}{4+\pi} \approx 1.401$ meter and the measure of the vertical dimension of the rectangular portion $\mathrm{y}_{max} = \frac{5(\pi+2)}{(4+\pi)} \approx 3.599$ meter.