

Answer on Question #44541 – Math - Algebra

Problem.

If the sum of the roots of a cubic equation is 3, the sum of the squares of the roots is 11 and the sum of the cubes of the roots is 27, find the equation. Also, solve the equation and find all its roots.

Solution.

If x_1, x_2, x_3 are the roots of the equation $x^3 + ax^2 + bx + c$, then by Vieta's formula

$$x_1 + x_2 + x_3 = -a, x_1x_2 + x_2x_3 + x_1x_3 = b, x_1x_2x_3 = c.$$

We know that $x_1 + x_2 + x_3 = 3, x_1^2 + x_2^2 + x_3^2 = 11, x_1^3 + x_2^3 + x_3^3 = 27$.

$$2x_1x_2 + 2x_2x_3 + 2x_1x_3 = (x_1 + x_2 + x_3)^2 - (x_1^2 + x_2^2 + x_3^2);$$

$$x_1^3 + x_2^3 + x_3^3 - 3x_1x_2x_3 = (x_1 + x_2 + x_3)(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3 - x_1x_3).$$

Therefore,

$$\begin{aligned} x_1x_2 + x_2x_3 + x_1x_3 &= \frac{(x_1 + x_2 + x_3)^2 - (x_1^2 + x_2^2 + x_3^2)}{2} = \frac{3^2 - 11}{2} = -1; \\ x_1x_2x_3 &= \frac{x_1^3 + x_2^3 + x_3^3 - (x_1 + x_2 + x_3)(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3 - x_1x_3)}{3} \\ &= \frac{27 - 3 \cdot (11 - (-1))}{3} = -3. \end{aligned}$$

Hence x_1, x_2, x_3 are the roots of the equation $x^3 + (-3)x^2 + (-1)x + 3 = 0$.

$$\begin{aligned} x^3 + (-3)x^2 + (-1)x + 3 &= x^3 - 3x^2 - x + 3 = x^2(x - 3) - (x - 3) = (x^2 - 1)(x - 3) \\ &= (x - 1)(x + 1)(x - 3). \end{aligned}$$

Since the roots of the equation $x^3 - 3x^2 - x + 3$ are $-1, 1, 3$.

Answer: $x^3 - 3x^2 - x + 3$ and $-1, 1, 3$.